THESIS

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"THE GENERAL MECHANISM OF EVAPORATION AND PRODUCTION OF SATURATED BRINE BY NATURAL EVAPORATION OF SEA WATER IN LARGE PONDS."

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I N D E X.

1.

SUMMARY.

The principle of manufacturing salt from the sea is discussed. An examination of the literature indicates that the only relevant investigation which has been carried out is on the rate of evaporation from large bodies of water. A summary of the relevant methods of assessing the evaporation is made and it is concluded that the aerodynamic approach is the most promising. An approximate equation describing the evaporation from Lake Grassmere as a function of time and brine concentration is derived for average weather conditions.

A qualitative discussion on the mechanism of natural evaporation of water from brine ponds by solar energy leads to energy balances which, it is suggested, should be completely investigated to produce an optimum depth of brine in a pond.

Material balances are set up for the general case of flow in n ponds. These can not be solved, for the variation of flow with time and a stepwise procedure is adopted to allow calculation of the output of saturated brine under average weather conditions for one, two, more than two, and an infinite number of ponds in a system. The latter calculations are based on what is proved to be an invalid assumption so that the results are erroneous. For the one and two pond cases, an estimate of output of saturated brine under average weather conditions is made and the time required to reach the stage of maximum output assessed.

2.

INTRODUCTION AND GENERAL PRINCIPLES.

Salt has been crystallised from the sea for many centuries. One of the early records of its manufacture from sea brines by solar evaporation is at Syracuse in 1790 (Ref. 1).

Since that time the output of salt and the number of places at which salt is produced by this method have greatly increased (Refs. 68, 69, 70, 71) until today, natural (or solar) evaporation of brine represents a major source of the chemical raw material - sodium chloride.

This chemical has many uses such as in the tanning, food and process industries, for glazing and as a raw material for the manufacture of:- Soda ash; caustic soda; sodium sulphate; sodium chlorate; sodium nitrate, chlorine and so on.

Compared with this great increase in production and utilisation, however, technical knowledge associated with the process has made little advance over the years. Indeed, what changes have occurred, have been largely due to what might be termed "empirical advances" - try something and see if it works - rather than an attempt to formulate an underlying mechanism on which to base investigations.

The present state of knowledge is thus rather limited and the aim of this report is to summarise the relevant knowledge to date and to initiate thought in such directions as might produce answers to the fundamental question of the mechanism of natural evaporation from large brine ponds, and quantitative assessment of the relative importance of the various factors which influence the evaporation and the consequent production of salt.

Around the New Zealand coast, sea water contains about 3.52% of dissolved salts (Ref. 60), the composition being given in table II/1.1

The average density of sea water is about 1.027 with a salinity (considered as pure Na Cl) of 3.72% (Ref. 39)

SALT	%
NaCl	2.787
$MgCl_2$	0.360
$MgSO_4$	0.238
$CaSO_4$	0.119
$CaCO_3$	0.014
KBr etc.	Trace
	3.518

TABLE II/1.3 - Composition of sea water around Lake Grassmere, N.Z.

On evaporating off water, the density of the brine increases and with it, the concentration of the dissolved salts increases. When the solution becomes saturated with respect to any one component, that salt begins to crystallise out as a solid, and continues to be so removed while water is evaporated off. On evaporating sea water, the brine solution becomes saturated with calcium sulphate first- [solubility of $CaSO_4$ = 2 X 10^{-1}gm of salt per 1000 c.c. of pure water at 20°C (Ref. 39a)] which begins to be deposited when the total dissolved solids content is about 27%; i.e. the brine density is about 1.13 or 16°Bé. (Ref. 29)

[Note, the density of a solution is usually denoted, in this work, by degrees Baùme, defined by

$$^{\circ}Bé = 145 - \frac{145}{S.G} - \quad (II/1\text{-}1) \quad \text{where}$$

S.G. = the specific gravity of the brine

$$= \frac{\text{density of the brine}}{\text{density of water at the same temperature.}}$$]

Deposition of calcium sulphate continues, and at a density of about 25°Bé, depending on the temperature, sodium chloride begins to crystallise out. [Solubility of NaCl at 20°C = 36 gm per 100 c.c. of pure water. (Ref. 39a)] At about 29°Bé, the brine becomes saturated with magnesium sulphate and then magnesium chloride.

[Solubilities at 20°C of:-

$MgSO_4$ = 30.8 gm/100 c.c. of saturated solution

$MgCl_2$ = 54.5 gm/100 c.c. of pure water . (Ref. 39a)]

In practice, the sea water is pumped into settling ponds to allow removal of solid impurities, such as sand, and then to a concentrating area where water is evaporated off under the influence of wind and energy received by solar radiation. (Ref. 70) This concentrating area is divided up into a number of ponds, and the brine flows, usually under gravity, from one pond to the next. When the brine reaches about 16°Bé, the calcium sulphate begins to crystallise out and settle on the bottom of the ponds. This deposit helps to seal the bottom against seepage of brine out, and if left undisturbed, a solid crust forms over the pond bottom. The brine continues to flow through the concentrating area until at about 25°Bé it is pumped into the crystallising area, where, with continuing evaporation, salt is deposited. At about 29°Bé the remaining liquor, called Bittern, is run off, for, although it still contains about 20% of the original salt (Ref. 1), it is now nearly saturated with magnesium salts which would crystallise out with the remaining NaCl. This impurity in the salt usually must be avoided, although, in some works, this bittern is run to further ponds and concentrated to 32°Bé (Ref. 1,70) when most of the NaCl is considered to have been removed. (Ref. 29) This latter deposit is useful as a raw material in some industries or can be purified for ordinary use by separating the magnesium salts from the NaCl by re-crystallisation.

More often, however, the Bittern is run to waste, despite its value as a source of magnesium, potash salts and bromine. A typical analysis of a Bittern containing 365.9 gm of solids per litre of the solution given in table II/1.2 (Ref. 1)

This solution could clearly be utilised as a source of these valuable materials. It is considered that the most economic way of extracting the valuable materials from sea water, would be to remove the magnesium compounds from the sea water before evaporation (Ref. 2) allow the $CaSO_4$ to be removed as before, and then evaporate the brine

until all the NaCl is removed. The remaining bittern would then be a source of potash salt and bromine.

SALT	% OF ANHYDROUS RESIDUE
KCl	7.6
NaCl	19.24
$MgCl_2$	46.77
$CaSO_4$	0.46
$MgSO_4$	25.00
$MgBr_2$	0.93

Table II/1.2 - A typical analysis of bitterns at the Oliver Salt Co.

The salt, exposed after running off the bittern, is harvested by hand or by machinery, washed and purified, if necessary, and graded according to demand.

Thus the technology of the production of salt by solar evaporation can be reduced to three important sections:

(i) The production of saturated brine from sea water by evaporation, and the influence on this evaporation of external factors such as wind, sun's incident radiation, air temperature and humidity, and the effect of inherent factors such as size, shape and number of ponds in a system of given total size, depth of brine in the pond and rates of flow of brine. These problems involve the mechanism by which water is evaporated from the brine by natural evaporation and the resulting energy and material balances over the system. The investigations which are the subject of this report were aimed at solutions to problems in this section.

(ii) The crystallisation of the salt from saturated brine and the evaluation of the factors which influence it.

(iii) Harvesting of the deposited salt from the pond bed with particular regard for the efficiency of lifting the salt, its transportation, washing and distribution and the effect of such harvesting on the mud bottoms of the ponds.

Sections (ii) and (iii) present numerous problems and there is room for a great deal of fundamental investigation in these two fields.

3.
PRODUCTION OF SATURATED BRINE.

This field of investigation is aimed at assessing the factors which will affect the output of saturated brine from a system of ponds into which sea water is pumped. The brine flows through the system and water is evaporated off, the amount of saturated brine produced being dependent on the rate of removal of water. Thus fundamental to this study of production of saturated brine is the mechanism of evaporation, determination of the evaporation rate and the energy requirements of evaporation. With this basis, material balances can be set up, residence times calculated and the yields of brine expressed in terms of known variables

§ 1.
The Mechanism of Evaporation:

The Kinetic theory of vapour pressure (Ref. 3) supposes that the molecules of the liquid and the vapour above it are in constant motion, and at a surface of the liquid some of the molecules will cross the surface into the vapour space above. If the conditions are such that an equilibrium state can not be set up, molecules will continue to leave the liquid and the liquid will evaporate. To ensure that evaporation continues, then, the vapour produced must be carried away from the liquid surface.

Further, as molecules in the vapour possess, on the average, more potential energy than those in the liquid, it is clear that the process of evaporation must be accompanied by an absorption of energy,

the latent heat of vapourisation. From this concept, evaporation depends upon a supply of energy and some mechanism of removing vapour from the space above the liquid.

The latter can be directly related to the intensity of turbulent mixing, itself related to wind velocity, and to a humidity gradient set up by this mixing process (Ref. 22). Thus external factors will determine the transport of moisture away, and little control can be placed upon these factors to produce greater or less evaporation.

The energy absorption, however, can be subject to control and it is important to investigate the variables, and assess their values, to determine the optimum working conditions.

In natural evaporation, assume that a sufficient mechanism is available to remove all the moisture evaporated from the body of the liquid so that the evaporation is dependent only on the energy balance.

Consider an infinite pond evaporating water to the air under the influence of the sun and the wind. (See figure 1.1)

If I_λ is the intensity of incoming radiation of wave length λ at such time that the angle of incidence is α, and if r_λ be the reflectivity of the water surface to radiation of wavelength λ incident at α, the energy which passes through the surface at such points as B is $(1 - r_\lambda)\ I_\lambda$. This energy is available for absorption into the body of fluid and so is initially available for the evaporation of water. α varies with the time of day and the time of the year; r with α (see table 1.0), λ and the intensity of waves on the surface which depend on u (Ref. 65); the total incident radiation has a spectral distribution, varying in wave length from about 300 $\mu\mu$ to 3000 $\mu\mu$ (Refs. 50, 51, 59, 62, 63) as shown in figure (1.2).

Hence the total energy absorbed in a finite time t, say, will be:-

$$I_t = \text{function (time, } \lambda\text{, u)}$$

$$= \Sigma_t \ I_\lambda \ (1 - r) \qquad - \qquad (1 - 1)$$

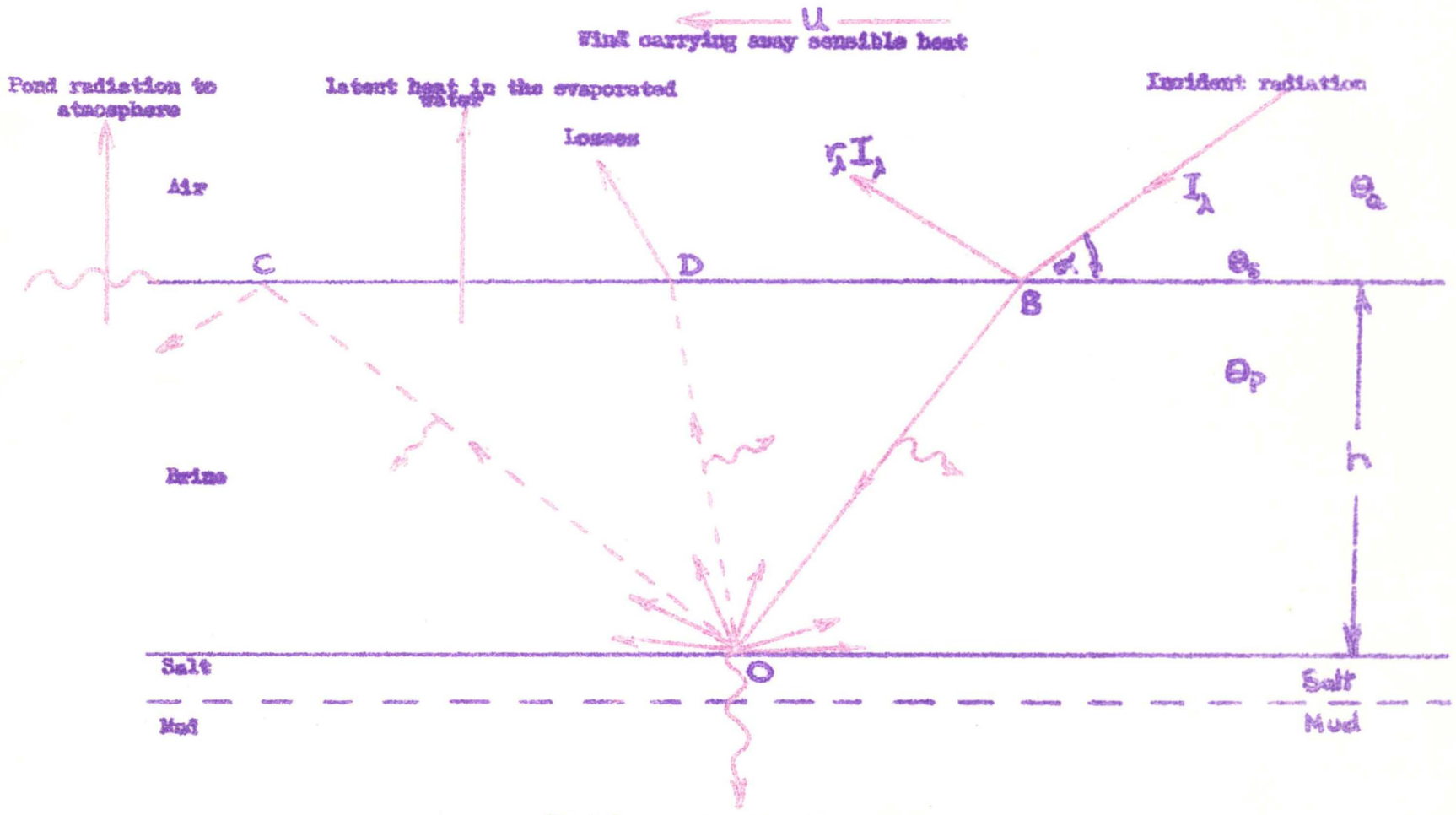


Fig. (1.1)

The absorption of Solar Radiant Energy in brine ponds.

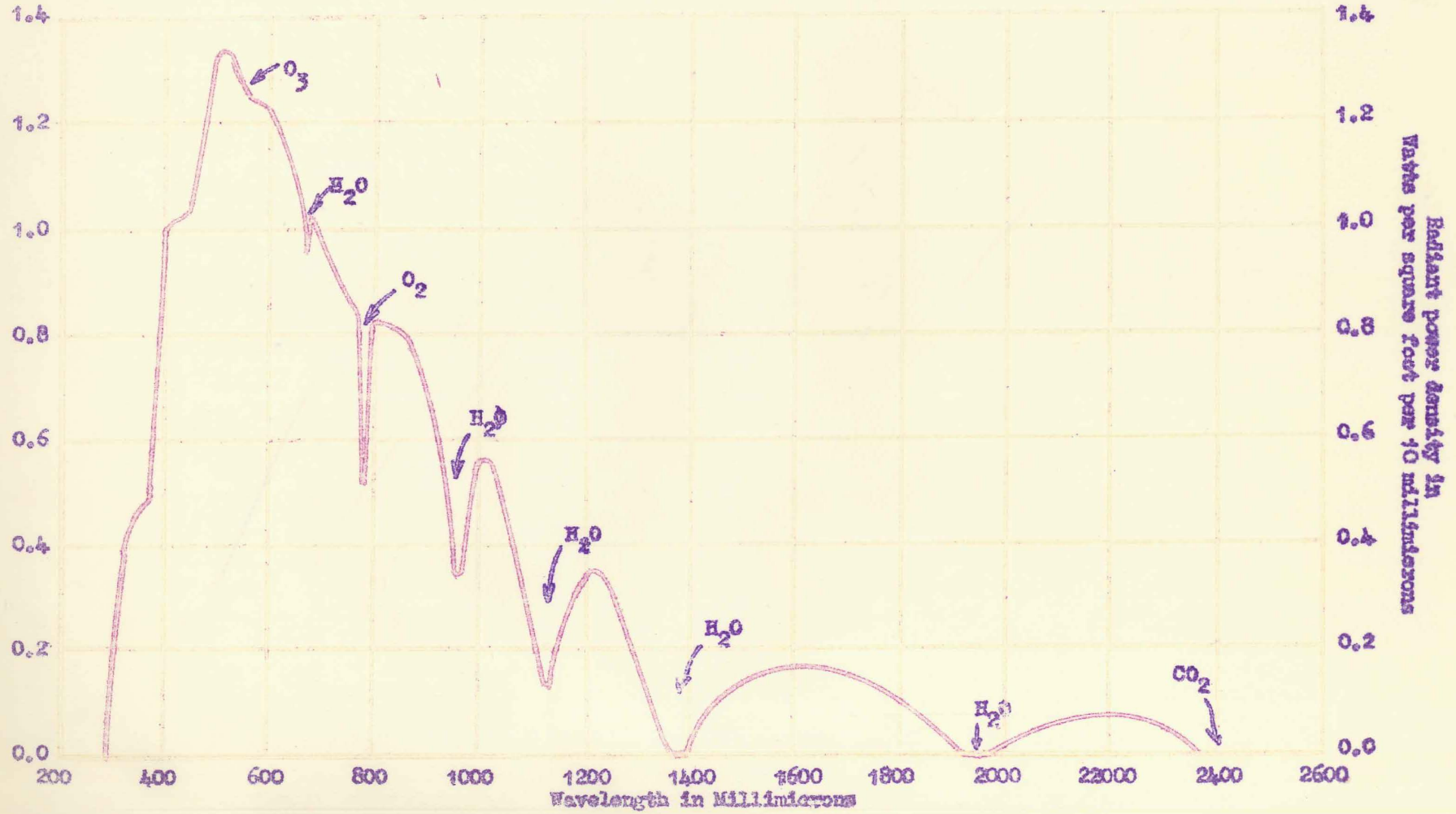


Fig. (1.2)
Distribution of Solar energy at the Earth's Surface.

Incident Angle	5°	10°	20°	30°	40°	50 - 90°
% Short wave radiation reflected.	40	25	12	6	4	3

Table 1.0 - Indicating the variation of reflectivity with incident angle (Ref. 25).

The evaluation of this function will be difficult as it involves an integration over time and wave length.

The energy wave, $(1 - r)\,\underline{I}_\lambda$ passes into the liquid and is partly absorbed as it is transmitted to the bottom of the pond. If h' is the path length BO, the energy absorbed, given by Lambert's law,

$$= \underline{I}_\lambda \ (1 - e^{-\mu h}) \quad - \quad (1 - 2)$$

where μ = co-efficient of absorption. (Ref. 52)

and is given by:-

$$\mu = \frac{4 \pi \eta \mu^1}{\lambda}$$

where η = refractive index of the media

λ = wave length of light

μ^1 = index of absorption or extinction co-efficient.

i.e. The value of μ depends on the nature and concentration of the fluid in the pond, the wave length of the energy and the brine temperature, so that the energy absorbed in the passage BO is a function of the incident radiation, the time, the wave length, wind velocity, depth of brine, concentration and nature of the pond fluid, and temperature of the brine.

Previous history of the brine will affect this absorption since the temperature of the brine affects the value of μ , which affects the energy absorbed, which controls the future brine temperature. Thus, continuous records and integration are necessary to evaluate the abosrbed energy.

On arriving at O, on the pond bottom, the energy could divide into three parts:-

(1) Part may be absorbed into the earth and lost for all time.

(2) Part may be absorbed in the top layers of the pond bottom and be re-radiated back into the liquid as long wave radiation which is quickly absorbed.

(3) The rest of the energy will be diffusely reflected from the pond bottom, particularly if a salt cake is present. An investigation by Free and Hoe, at Canterbury College (Ref. 49) on a diffuse reflection of radiant energy from a salt cake in air, showed general agreement with results given in International Critical tables (Ref. 74) for other crystal reflecting surfaces. The theoretical approach by Ornstein and van de Burg (Ref. 53) did not appear to be satisfactory and the general conclusion was that to first order, the cosine distribution law could be used regardless of the angle of incidence. At least the cosine law gives the limit to which the reflection tends, and would be satisfactory for calculating the distribution of energy from salt crystals on the bottom of a salt pond. It appears that the refractive index of the medium above the salt cake would not seriously affect the distribution. The energy which is reflected is transmitted through the liquid again, being partly absorbed as it goes, there being two important parts of the energy. That which is reflected from the bottom so that it subtends an angle at the surface greater than the critical angle and that less than the critical angle (which depends on the refractive index of the solution η). That energy which arrives at the surface at an angle which is less than the critical angle will pass through the surface and be lost, while that at an angle greater than critical can be reflected back into the solution or pass on out, the relative amounts depending on the reflectivity of the underside of the surface r'. r' may or may not equal r . An intergration over time, wavelength and angle of the energy diffusely reflected from the

bottom which passes through the surface would be required to give the loss of energy by this manner. That energy which is not lost at the surface will, of course, continue this series.

This is the mechanism by which the energy is absorbed. It is clearly of a very complex nature and any solution will be difficult (Ref. 33). This absorbed energy is then available for the various losses and for evaporating water. To get the maximum evaporation of water, the maximum energy must be absorbed and the minimum energy lost by conduction to the pond bottom and sides, by radiation to the atmosphere and by convection to the air by wind movement.

An optimum depth, h_{opt}, thus suggests itself so that a minimum energy reaches the surface within the critical angle and hence the maximum is absorbed. This optimum depth can not be simply calculated from Lambert's law (Ref. 52) since for zero transmission of energy:-

$0 = I\ e^{-\mu h'_{opt}}$, would require an infinite path length h'_{opt} and hence an infinite pond depth h, but would be calculable from a solution of the energy absorbed. The depth should not be made too great, as the greater the depth, the more energy which must be utilised as sensible heating to raise the pond temperature. Hence there will be a balance and an optimum depth.

Other factors can influence this. Wave Formation on the pond surface has been mentioned. The cloudiness of the sky and concentration of dust can have great effect on the intensity of the short wave radiation reaching the pond (Ref. 54). Clouds will absorb the solar energy and re-radiate it as long wave radiations.

Thus, generally, a solution of the problem of the energy absorbed into the body of liquid in a pond, is dependent on evaluating and relating the many factors and variables: The intensity of incoming radiation at different wave lengths which is a function of time and cloudiness of the sky; the reflectivities of the surface for the various wave lengths under rippled surface conditions; the absorption of energy into the brine which depends on the wave length, brine density, depth of brine, and the intensity and angle of the incident radiation; the reflectivity and nature of the reflection at the pond bottom; internal

reflection at the underside of the surface and the critical angle reflection; the general loss terms of conduction, radiation and convection. Such a relation must be integrated over the wave length range and time range to evaluate the total absorbed energy.

§ 2.

An energy balance:

Energy balances have been discussed fully by Ferguson (Ref. 25), Block et al (Ref. 33), Sheppard (Ref. 6), and others.

The energy absorbed into the brine (See § 1) can be dissipated in a number of ways:

(i) The pond, being at a temperature θ_p , can radiate to the atmosphere at a rate per unit area of:-

$$\sigma\, \varepsilon_p (\theta_p + 460)^4 - \varepsilon_{at}\, \alpha_{p_{\theta_a}} (\theta_a + 460)^4 \qquad - (2 - 1)$$

where ε_p = emissivity of the pond surface

ε_{at} = emissivity of the atmosphere

$\alpha_{p_{\theta_a}}$ = absorptivity of the pond surface

to radiation of such wave length as is radiated from the atmosphere at temperature θ_a .

(ii) The temperature of the pond can vary in any given time due to sensible heating at the rate of:-

$$h \rho\, S_p \frac{d\theta_p}{dt} \qquad - (2 - 2)$$

where h = depth of brine in the pond

ρ = density of the brine

$\frac{d\theta_p}{dt}$ = rate of change of θ_p w.r.t. time

S_p = specific heat of the fluid between the limits of the variation of θ_p .

(iii) Heat energy can be transferred from the pond to the air mass above it by convective heat transfer at the rate:-

$$h_c \left[\theta_{p_s} \sim \theta_a \right] \quad - \quad (2 - 3)$$

where h_c = heat transfer coefficient for transfer of heat by convection from the pond surface. Much work has been done on evaluating h_c for such a situation as this. (Typical references 25, 26, 55, 56, 57). Fishendon (Ref. 55) claims:-

$$h_c = 0.48 + 0.083\ u \quad \text{cae/sq cm/hr/}^{\circ}\text{c}$$

for forced convection heat transfer from a plane surface, to a gas flowing parallel to it is a satisfactory relation. Generally, h_c is given as some function of the wind velocity u and should be determined experimentally over the pond surface for each particular area, since the height at which the velocity is measured, the type of wind structure and the nature of the pond surface and surrounding ground affect the value of h_c. (Ref. 25)

The term θ_{p_s}, the temperature of the surface of the pond, is an impractical term to measure and the final expression for the total energy balance must be solved for θ_{p_s} by substituting an expression for E (see later). Many workers have confused the surface pond temperature with the bulk pond temperature, θ_p, and absorbed the error in the expression for h_c $\left(\theta_p \sim \theta_a \right)$ (Refs. 33, 20)

This is clearly an erroneous simplification since h_c, θ_{p_s} and θ_p are connected only in a very complex manner and not merely by simple constants. It remains that θ_{p_s} is an unknown in the heat balance equation and must be solved for rigorously.

(iv) The latent heat of vapourisation of water, L, must be

provided per unit weight of water evaporated, so for a weight m of water evaporated, the heat required will be mL heat units.

(v) Such terms as heat of crystallisation of the NaCl, heat of concentration of brine and the heat change in the salt are all small and may be neglected.

(vi) It has often been assumed that the heat loss from the pond bottom is negligible (Ref 25, 28). This heat loss is by conduction through the soil

$$= k_s \frac{\Delta \theta g}{L} \quad - (2 - 4)$$

Where k_s = the co-efficient of heat transfer by conduction for the soil in the temperature range involved

$\frac{\Delta \theta g}{L}$ = temperature gradient in the ground.

Average measurements taken at Lincoln College in February, 1954, of soil temperatures at depths of 4", 8", 12", 36", are plotted on the curve (fig. 2.1). From this curve, an average temperature gradient of $\frac{1}{7}$ °F is observed. A value of k_s was estimated from data available in Perry (Ref. 56) at 0.2 BTU/ft^2/ft/hr/°F

Hence the heat flow into the ground

$$= \frac{0.2}{7} \times 12 \text{ BTU/hr/ft}^2$$

$$= 0.34 \text{ BTU/hr/ft}^2$$

In new Zealand the average energy arriving on a horizontal surface from the sun per day, is of the order of 1000 BTU/sq. ft. for an average of about $5\frac{1}{2}$ hours sunshine. (Ref. 59)

i.e. energy = 180 BTU/sq. ft./hr.

On the average about 7% of this is lost by reflection from a water surface (Ref. 59) and if it is assumed a third of the energy is absorbed on the first passto the pond bottom, then energy striking the bottom = 110 BTU/hr/ft^2

Thus the loss by conduction $\sim$ 0.3%

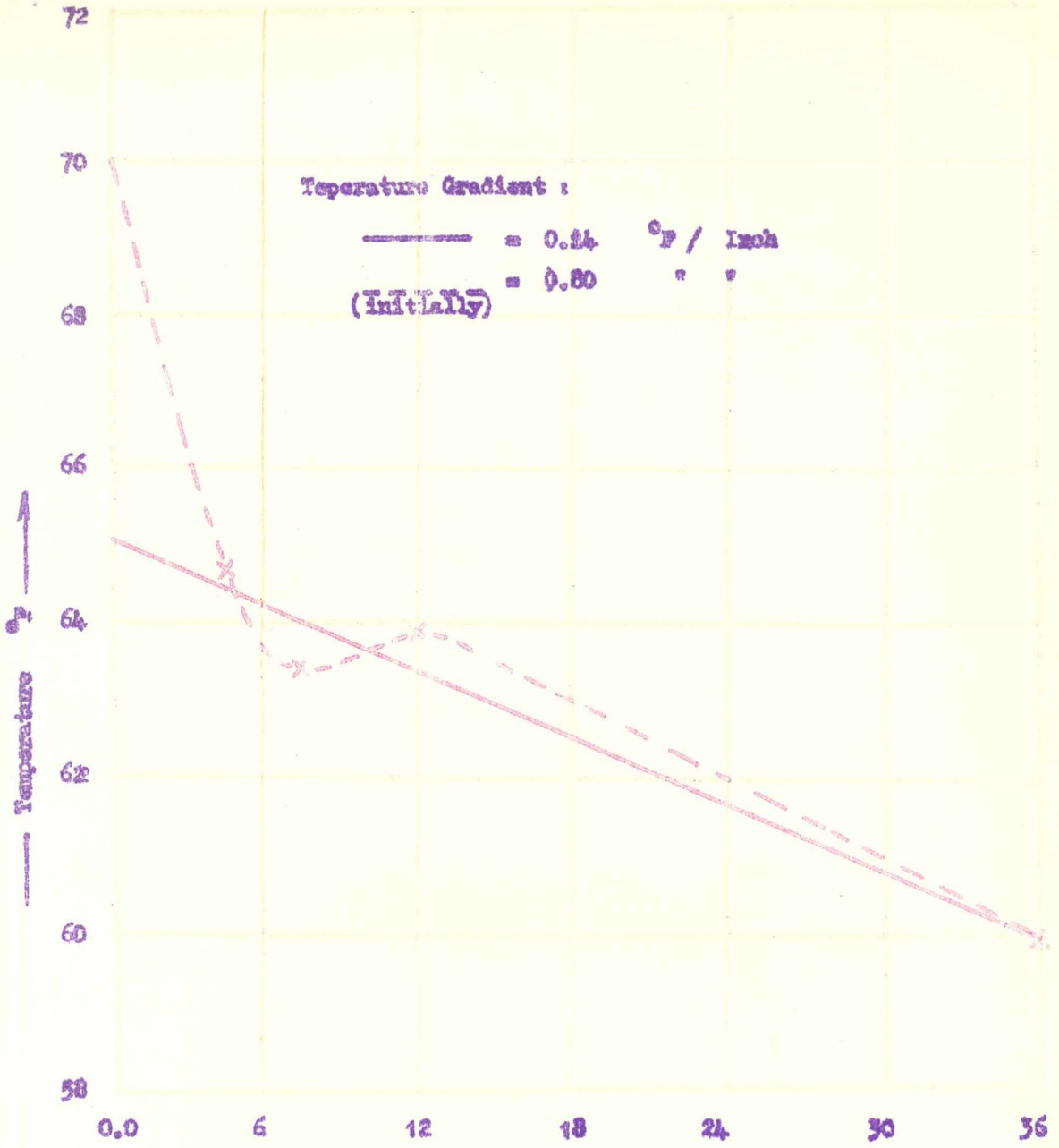


Fig. (2.1) Temperature gradients in the soil as measured at Lincoln College in February 1954.

Thus, considering the pond bottom as a heat insulator involves an error $\sim$ 0.3% which generally could be neglected.

On summation of the energy terms, a heat balance gives per unit area per unit time:-

$$\frac{I}{\Delta t} \text{ absorbed} = \pm h_c \left[\theta_{p_s} - \theta_a\right] \pm \sigma \left[\varepsilon_p (\theta_p + 460)^4 - \varepsilon_{at} \alpha_{p\theta_{at}} (\theta_a + 460)^4\right]$$

$$\pm \rho_p \; h \; s_p \left[\frac{\Delta \theta_p}{\Delta t}\right] \pm k_s \frac{\Delta \theta}{l} g \pm L \frac{m}{\Delta t}$$

where Δt = the time interval - (2 - 5)

$\Delta \theta_p$ = change in the pond temperature in time Δt

or instantaneously in time δt, as $\delta t \rightarrow o$ neglecting the loss from the pond bottom,

$$\frac{dI}{dt} \text{ absorbed} = \pm h_c \left[\theta_{p_s} - \theta_a\right] \pm \sigma \left[\varepsilon_p (\theta + 460)^4 - \varepsilon_{at} \alpha_{p\theta_{at}} (\theta_a + 460)^4\right]$$

$$\pm \rho \, hs_p \frac{d\theta_p}{dt} \pm L \frac{dm}{dt} \quad \text{--- (2 - 6)}$$

$\frac{dm}{dt}$ can be replaced by the instantaneous value of evaporation rate E, so that given the other terms, equation (2 - 6) is an expression relating E and θ_{p_s} which can be solved for θ_{p_s} if another independent expression for E can be used. (See § 3, 2)

An optimum depth of brine is given for maximum absorption of energy and maximum utilisation of that energy for evaporation, so that solution of equation (3 - 6) for optimum depth conditions would give the maximum E .

§ 3.

Evaporation Rates:

This section of the problem - the estimation of the rate of removal of water from large areas - has been extensively investigated.

Bonython (Ref. 4) has discussed the various meanings which can be associated with evaporation rate, and in general, it can be defined as the rate of conversion of water from a liquid phase to water vapour in a vapour phase. In this case, it is the rate of removal of water from a solution held in a large pond to the atmosphere above the pond.

A further necessary definition is of nett evaporation equal to the total evaporation from a pond surface, minus the rainfall on that same area, all expressed over the same period of time. It is the nett evaporation which has been largely used in "Solar salt" calculations. The usual figures quoted (Ref. 11) are nett evaporation from ponds containing pure water expressed as averages over a period of a year. These figures clearly would not apply to salt solutions under the same conditions due to the depression of the vapour pressure of a solvent (water) by addition of a solute. (Ref. 3) and (Ref. 30)

viz: $$\frac{p^o - p}{p^o} = \frac{n_2}{n_1 + n_2} = X_2$$

where X_2 = mole fraction of the solute in the solution

p^o = v.p. of the pure solvent

p = v.p. of the solution

Clearly the depression of the vapour pressure increases as the mole fraction of the solute increases, so that evaporation would be expected to decrease in some way as the concentration of brine increased. Despite this, values of nett evaporation from pure water have been extensively used in solar salt works design. Typical values are shown in table (3.1) (Ref. 11)

PLACE	Average Rainfall inches/year.	Average Grass evap. inches/year.	Average Nett evap. inches/year.
Grassmere, N.Z.	24.13	49.48	25.35
Cheetham, Australia.	19.98	47.78	27.80
Newark, California.	8.10	58.2	ca.48

Table (3.1) - Typical values of rainfall and evaporation from free water surfaces.

From what has been stated above, brines varying in concentration from sea water to about 30° Be' will not evaporate 25" of water at Lake Grassmere in an average year. It is clearly necessary to establish nett evaporation for different concentration brines and to note the way this evaporation varies with time.

This aspect itself has received very little attention, due to the unsatisfactory state of knowledge about methods of practically estimating E with any reliability or accuracy. Bonython (Ref. 23) is now attempting to apply established methods to brine ponds at Salt Creek. Consequently, the great deal of work which has been done, has been directed to estimating evaporation rates from pure water in large ponds and Cummings (Ref. 72) has summarised the work. Generally, there have been three main approaches.

(i) Measurement of evaporation directly from large ponds or expanses of water.

(ii) Measurement of evaporation in small tanks, or evapormeters, and the correlation of these values with the evaporation from large ponds.

(iii) The calculation of evaporation from other more easily measurable physical properties such as air temperature, pond temperatures, air humidities, wind velocities and other meteorological data.

The existing literature has been studied and an extensive summary of the present state of knowledge of evaporation is given in an appendix to this report.

Because of the unsatisfactory state of knowledge on this subject when applied to evaporation from brine ponds and the absence of complete data from Lake Grassmere, an estimate of evaporation was made.

Estimated evaporation data for Grassmere.

Ellis, (Ref. 14 & 28) carried out an investigation at Lake Grassmere in the months of January and February, 1953, and determined an average experimental curve of the ratio F, equal to (Gross evaporation from brine in a pond) to (gross evaporation from water in an evapormeter) against the concentration of the brine for that period. This is reproduced in Fig (3.1). It has been assumed that these ratios can be taken for all the summer months, but that lower factors will apply in the Winter months, April to August inclusive. Ellis also estimated that for the winter period, the ratio would be 0.52 for 25° Be'. On studying the data for Ellis' curve, it is clear that the average curve results from a number of such curves of similar shape so that if the mechanism governing this ratio is constant, it is feasible to suggest the winter and summer curves should be parallel. It is assumed the curves apply throughout the entire period, that is the ratio for any concentration is constant during the summer and changes in the winter to another constant value.

Both curves are given in Figure (3.1). The evaporation can then be calculated for any particular brine concentration at any particular time by:

$$\text{Average nett evaporation from brine of concentration X} = \left\{ \text{average gross evaporation from pure water in an evaporimeter} \right\} \times \left(\text{Evaporation factor for the particular time and concentration X} \right) - \left\{ \text{Average rainfall at that time.} \right\} \quad - (3 - 16)$$

(NOTE - Equations (3-1) to (3-15) appear in the appendix to § 3)

The average rainfall and gross evaporation of pure water in an evaporimeter tank for Grassmere were available for six years only (Ref. 48). Table (3-10) gives the results of applying equation (3-16) to this data, and a plot was made of the nett average evaporation for twelve months against concentration of brine (Fig. 3.2)

Fig. (3.1)

Graph of the ratio of the predicted evaporation rates from brine ponds to fresh water evaporimeters.

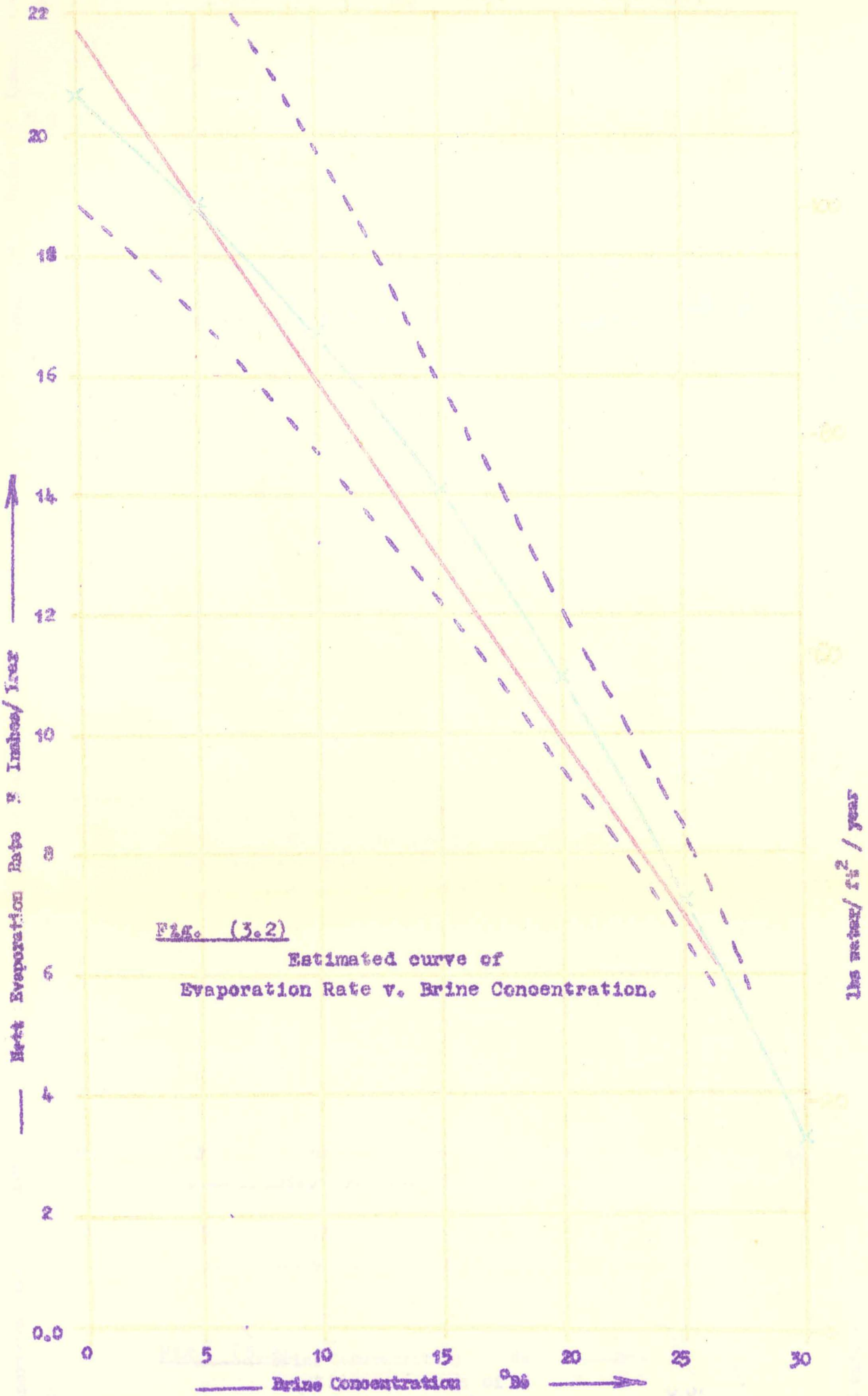


Fig. (3.2)

Estimated curve of
Evaporation Rate v. Brine Concentration.

INCHES EVAPORATION PER MONTH.

MONTH	FRESH WATER EVAP. AV. TANK VALUES.			0°Bé			5°Bé			10°Bé			15°Bé			20°Bé			25°Bé			30°Bé		
	Gross	Rain	Nett	F Factor	Gross evap. from Brine	Nett evap. from Brine	F	E_g	E_n	F	E_g	E_n	F	E_g	E_n	F	E_g	E_n	F	E_g	E_n	F	E_g	E_n
JAN.	6.91	2.20	$4^{+}.71$	0.95	6.56	4.36	0.91	6.28	4.08	0.87	6.01	3.81	0.82	5.66	3.46	0.76	5.25	3.05	0.68	4.70	2.50	0.60	4.14	1.94
FEB.	5.86	1.25	$4^{+}.61$	0.95	5.56	4.31	0.91	5.33	4.08	0.87	5.09	3.84	0.82	4.80	3.55	0.76	4.45	3.20	0.68	3.98	2.73	0.60	3.52	2.27
MAR.	5.36	1.21	$4^{+}.15$	0.95	5.08	3.87	0.91	4.87	3.66	0.87	4.71	3.50	0.82	4.39	3.18	0.76	4.07	2.86	0.68	3.64	2.43	0.60	3.21	2.00
APR.	3.35	2.10	$1^{+}.25$	0.80	2.68	0.58	0.76	2.55	0.45	0.70	2.35	0.25	0.65	2.18	0.08	0.58	1.94	$0^{-}.16$	0.52	1.74	$0^{-}.36$	0.45	1.51	$0^{-}.59$
MAY	2.28	2.53	$0^{-}.25$	0.80	1.82	$0^{-}.71$	0.76	1.73	$0^{-}.80$	0.70	1.60	$0^{-}.93$	0.65	1.48	$1^{-}.05$	0.58	1.32	$1^{-}.21$	0.52	1.19	$1^{-}.34$	0.45	1.03	$1^{-}.50$
JUNE	1.76	2.67	$0^{-}.91$	0.80	1.41	$1^{-}.26$	0.76	1.34	$1^{-}.33$	0.70	1.23	$1^{-}.44$	0.65	1.14	$1^{-}.53$	0.58	1.02	$1^{-}.65$	0.52	0.92	$1^{-}.75$	0.45	0.79	$1^{-}.88$
JULY	1.65	2.46	$0^{-}.81$	0.80	1.32	$1^{-}.14$	0.76	1.26	$1^{-}.20$	0.70	1.16	1.30	0.65	$1^{-}.07$	1.39	0.58	0.96	$1^{-}.50$	0.52	0.86	$1^{-}.60$	0.45	0.74	$1^{-}.72$
AUG.	2.20	2.61	$0^{-}.41$	0.80	1.76	$0^{-}.85$	0.76	1.67	$0^{-}.94$	0.70	1.54	$1^{-}.07$	0.65	1.43	$1^{-}.18$	0.58	1.25	$1^{-}.36$	0.52	1.14	$1^{-}.47$	0.45	0.99	$1^{-}.62$
SEP.	3.27	1.26	$2^{+}.01$	0.95	3.11	1.85	0.91	2.98	1.72	0.87	2.84	1.58	0.82	2.68	1.42	0.76	2.48	1.22	0.68	2.22	0.96	0.60	1.96	0.70
OCT.	4.58	2.25	$2^{+}.33$	0.95	4.35	2.19	0.91	4.17	1.92	0.87	3.98	1.73	0.82	3.76	1.51	0.76	3.48	1.23	0.68	3.12	0.87	0.60	2.75	0.50
NOV.	5.46	2.36	$3^{+}.10$	0.95	5.19	2.83	0.91	4.97	2.61	0.87	4.75	2.39	0.82	4.48	2.12	0.76	4.15	1.79	0.68	3.71	1.35	0.60	3.28	0.92
DEC.	6.71	1.60	$5^{+}.11$	0.95	6.37	4.77	0.91	6.10	4.50	0.87	5.84	4.24	0.82	5.50	3.90	0.76	5.10	3.50	0.68	4.55	2.95	0.60	4.02	2.42
Ann. Total	49.39	24.50	24.89		45.21	20.71		43.25	18.75		41.10	16.60		38.57	14.07		35.47	10.97		31.77	7.27		27.94	3.44
AVERAGE			2.07			1.73			1.56		1.38				1.17			.91			.61			.29

TABLE (3-1) Monthly average evaporations for brines of various concentrations evaporating in large ponds.

To check the reliability of these estimated values of evaporation, a comparison was made with the little actual data available from Grassmere. This data was tank evaporation for brines and itself is rather unreliable as the concentrations were not maintained constant over the periods of measurement. To adjust these readings to pond evaporations let:

$$F = \frac{\text{pond evaporation of brine}}{\text{tank evaporation of water}}$$

$$G = \frac{\text{tank evaporation of brine}}{\text{tank evaporation of water}}$$

$$\frac{F}{G} = \frac{\text{pond evaporation of brine}}{\text{tank evaporation of brine}}$$

$\therefore$ "measured" pond evaporation of brine $= \frac{F}{G}$ (measured tank evaporation of same concentration brine) - - (3 - 17)

A typical example is for 25° Be' brine for February, 1953:

Gross evaporation from brine in an evaporimeter (measured) = 2.33 inches

Gross evaporation from water in an identical evaporimeter (measured) = 5.03 inches

$\therefore$ $G = \frac{2.33}{5.03} = 0.465$

From figure (3.5), for February and 25° Be' brine $F = 0.68$

$\therefore$ Gross evaporation of 25° Be' brine from the pond "measured" = 3.45 inches

Estimating this same quantity by using the method of equation (3-16), a value of 3.98 inches is obtained, a difference of 14.5% of the "measured" value. Table (3 - 2) summarises the comparison.

From this table it appears that an uncertainty of about 15% exists with the values of evaporation estimated by equation (3-16). This uncertainty, defined now as the percentage probable error in E, may be due to the method of calculating by equation (3-16), or to errors in the evaporation of water or to errors in the "measured" values of evaporation for brines, or to errors in the estimated values of F. It should also be noted that the method of equation (3-16) gives average values whereas many of the "measured" values were taken

TABLE (3 - 2)

DATE	Supposed Density ^{o}Be	Measured Gross Evap. for Brine in Tank E_1 inches per Month	Measured Gross Evap. for Water in Tank E_2 inches per month	$G = \frac{E_1}{E_2}$	Estimated F from Curve Fig. 3.5	$\frac{F}{G}$	"Measured" Gross Evap. of Brine in Pond Eqn. (3-17) $E_{meas.}$	Average Gross Evap. of Water in Tank (Ref inches	Average Rainfall inches	Estimated Gross Evap. of Brine in Pond Eqn. (3-16) $E_{est.}$	$\frac{E_{est} - E_{meas}}{E_{meas.}}$ %	REMARKS. i.e. weather conditions etc.
FEB. 1954	26	3.50	7.203	0.486	0.65	1.34	4.70	5.86	1.25	3.9	- 17.0	Weather conditions above average conc. constant
JAN. 1953	23	4.15	6.85	0.606	0.70	1.16	4.80	6.91	2.20	5.0	+ 4.2	Nearly average - slightly below
FEB. 1953	23	3.06	5.01	0.609	0.70	1.15	3.52	5.86	1.25	4.2	+ 20.0	Weather little below average. Rather great variation in measured conc.
MARCH 1953	21.5	3.01	5.13	0.587	0.70	1.20	3.59	5.36	1.21	3.9	+ 8.4	Nearly average
MAY 1953	19.5	0.94	2.297	0.43	0.58	1.35	1.27	2.28	2.53	1.3	0.0	Average conditions
JUNE 1953	16	1.18	1.46	0.81	0.60	0.74	0.88	1.76	2.67	1.1	+ 34.0	Below average. Concentration of Brine varied greatly in this month. Result of no value.
FEB. 1953	25	2.33	5.013	0.465	0.68	1.46	3.45	5.86	1.25	4.0	+ 14.5	Below average.
MARCH 1953	25	2.29	5.126	0.446	0.68	1.53	3.50	5.36	1.21	3.64	+ 2.9	Little below average.
MAY 1953	18.5	1.14	2.29	0.497	0.60	1.21	1.38	2.33	2.53	1.36	0.0	Weather conditions average. Brine concentration constant.

under conditions far from average in specific years. Where average conditions apply and brine concentrations are reliable, reasonable agreement between the results is obtained, so that an unreliability of 15% is reasonable and probably generous.

This error "envelope" has been imposed on the figure (3.2) to indicate the range of uncertainty. In the worst possible case, for no value of X is there an average nett evaporation rate for a complete year which is negative, although from table (3.2) it is seen that E is negative for some months of the year.

From the plot of Figure (3.2) average nett evaporation for one year against concentration of brine, it is clear that within the limits of the allowable error, a straight line can replace the curve, so that at any given time:-

$$\left(\frac{\partial E}{\partial X}\right)_t = - \mu(t) \qquad \text{---- (3 - 18)}$$

Where E = evaporation rate

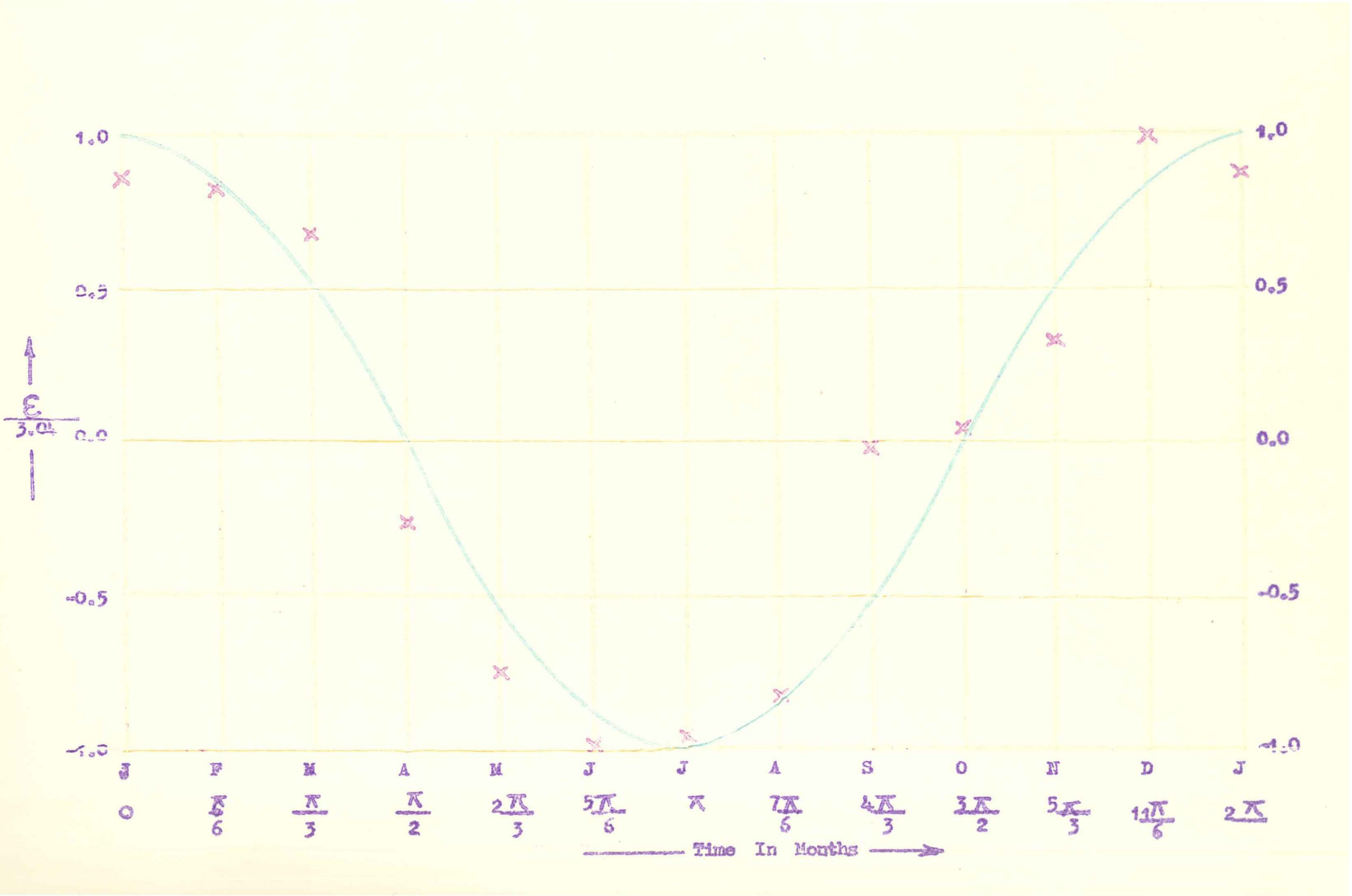
X = concentration of brine, stated as a fraction -

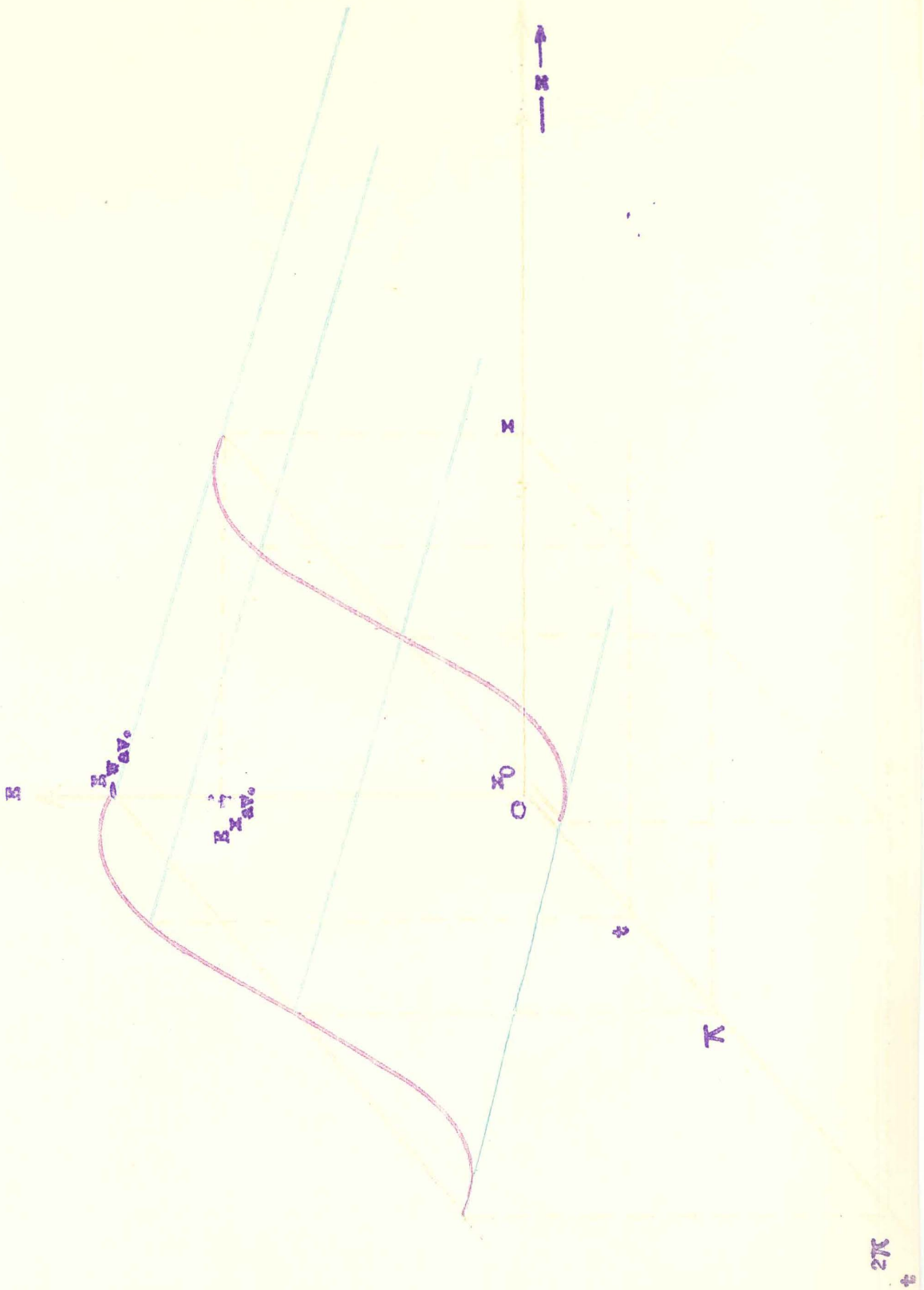
$$\frac{\text{Weight of NaCl salt in a sample of the brine}}{\text{Total weight of the brine sample}}$$

$\mu(t)$ = constant depending on the time.

Values calculated from this expression will apply only at some given time (at which $\mu(t)$ applies) and it is clearly necessary to investigate how the evaporation rate varies with time at any given concentration.

As all the values of evaporation in table (3.5) are based on the evaporation from pure water in an evaporimeter, the variation of this quantity with time will indicate the way the evaporation rates for all the different concentrated brines will vary. Table (3.3) gives the relevant figures plotted on figure (33). On this plot is also plotted a true sine curve and within the experimental uncertainty of the measured values, it can be written:-


1.0
0.5
0.0
-0.5
-1.0
ε / 3.0
J F M A M J J A S O N D J
0 π/6 π/3 π/2 2π/3 5π/6 π 7π/6 4π/3 3π/2 5π/3 11π/6 2π
Time In Months



$E_x = \lambda(x) \sin t + C_1$

$E_t = C_2 - \mu(t) \cdot x$

Fig. (3.4) The sine surface of (E, x, t)

$$\left(\frac{\partial E}{\partial t}\right)_x = \lambda(x) \text{ cosine } t \qquad \text{-----(3-19)}$$

Where $\lambda(x)$ = a constant whose value depends on the brine concentration x. With $\lambda(x)$ decreasing as x increases.

The equations (3-18) and (3-19) must be combined to give a relation of

$$E = f(x,t) \qquad \text{--- (3-20)}$$

From (3 - 19), at constant x,

$$E_x = \lambda(x) \sin t + c, \qquad \text{---(3-21)}$$

From (3 - 18), at constant time,

$$E_t = c_2 - \mu(t)\ x \qquad \text{---(3-22)}$$

So that E for any concentration and time is a point on a sine surface. This surface is shown qualitatively in figure (3.4)

Generally, there are two expressions then:

$$E = \lambda(x) \sin t + X(x) + C_3 \text{ from (3-21)} \qquad \text{---(3-23)}$$

and $$E = \tau(t) - \mu(t).\ x + C_4 \text{ from (3-22)} \qquad \text{---(3-24)}$$

Where $\lambda(x)$, $X(x)$ are functions of x only

$\tau(t)$, $\mu(t)$ are functions of t only

These two expressions must be identical, so that from (3-24), x must appear in (3-23) as x only (and not x^n where n has any value other than 1) i.e. generally in (3-23)

$X(x) \equiv (px + q')$ where p and q' are constants, and similarly

$\lambda(x) = (ax + b)$, a,b constants

Hence $E \equiv (ax + b) \sin t + (px + q') + C_3$

$= (ax + b) \sin t + (px + q)$ when C_3 is absorbed in q' so

$q = C_3 + q'$

To check, from equation (3-23), t must appear in (3-24) as sin t only and as no other function of t

MONTH	NETT EVAP. FOR WATER E	$\varepsilon = E - E_{av}$	$\varepsilon / 3.04$
JAN.	+4.71	+2.64	+0.864
FEB.	+4.61	+2.54	+0.836
MARCH	+4.15	+2.08	+0.685
APRIL	+1.25	-0.82	-0.270
MAY	+0.25	-2.32	-0.764
JUNE	-0.91	-2.98	-0.980
JULY	-0.81	-2.88	-0.949
AUG.	-0.41	-2.48	-0.817
SEPT.	+2.01	-0.06	-0.019
OCT.	+2.33	+0.26	+0.086
NOV.	+3.10	+1.03	+0.339
DEC.	+5.11	+3.04	+1.00
AVERAGE = +2.07			

Table (3.3) - Values for figure (3.3)

so $\tau(t) \equiv (l \sin t + m')$

and $-\mu(t) \equiv (g \sin t + k)$

where l, m', g, k are constants

Hence $E = (l + gx) \sin t + (m' + kx) + C_4$

$= (l + gx) \sin t + (m + kx)$ --- (3-26)

Equations (3-25) and (3-26) must be identical, and on comparison,

$E = (l + gx) \sin t + (m + kx)$

$= (b + ax) \sin t + (q + px)$

since a, b, p, q, l, g, m and k are all arbitrary constants

Hence the general equation for evaporation rate is:-

$$E = (ax + b) \sin t + (px + q) \quad \text{--- (3-27)}$$

$$\therefore \quad \frac{dE}{dt} = (a \sin t + p) \frac{dx}{dt} + (ax + b) \cos t \quad \text{---(3-28)}$$

where a, b, p, q are arbitrary constants

A time basis must be defined for equation (3-27)

For 1 cycle $t = 2\pi \equiv$ 1 year

Hence 1 month $\equiv \frac{\pi}{6}$

From the data of table (3.3)

For the month of December, $\varepsilon = E - E_{av}$

$= 3.04$ in/month

and for June $\varepsilon = -2.99$

$\therefore \quad \varepsilon_{av} = \pm 3.02$

Define December as $t = \frac{3\pi}{6}$ when ε = maximum = + 3.02, so t = 0 refers to the month of September, $t = \pi$ is March, $t = \frac{3\pi}{2}$ is June etc. (Fig 3.5)

For time intervals of $\frac{\pi}{6}$, the average monthly evaporation will be assumed to be at a time in the middle of the month, as shown in figure (3.5) for October, with maxima and minima at $t = \frac{\pi}{2}$ and $\frac{3\pi}{2}$ (December and June) respectively.

For equation (3-27), when t = o, E = px + q = annual average value of E for brine of concentration x.

For pure water, x = o, so that:

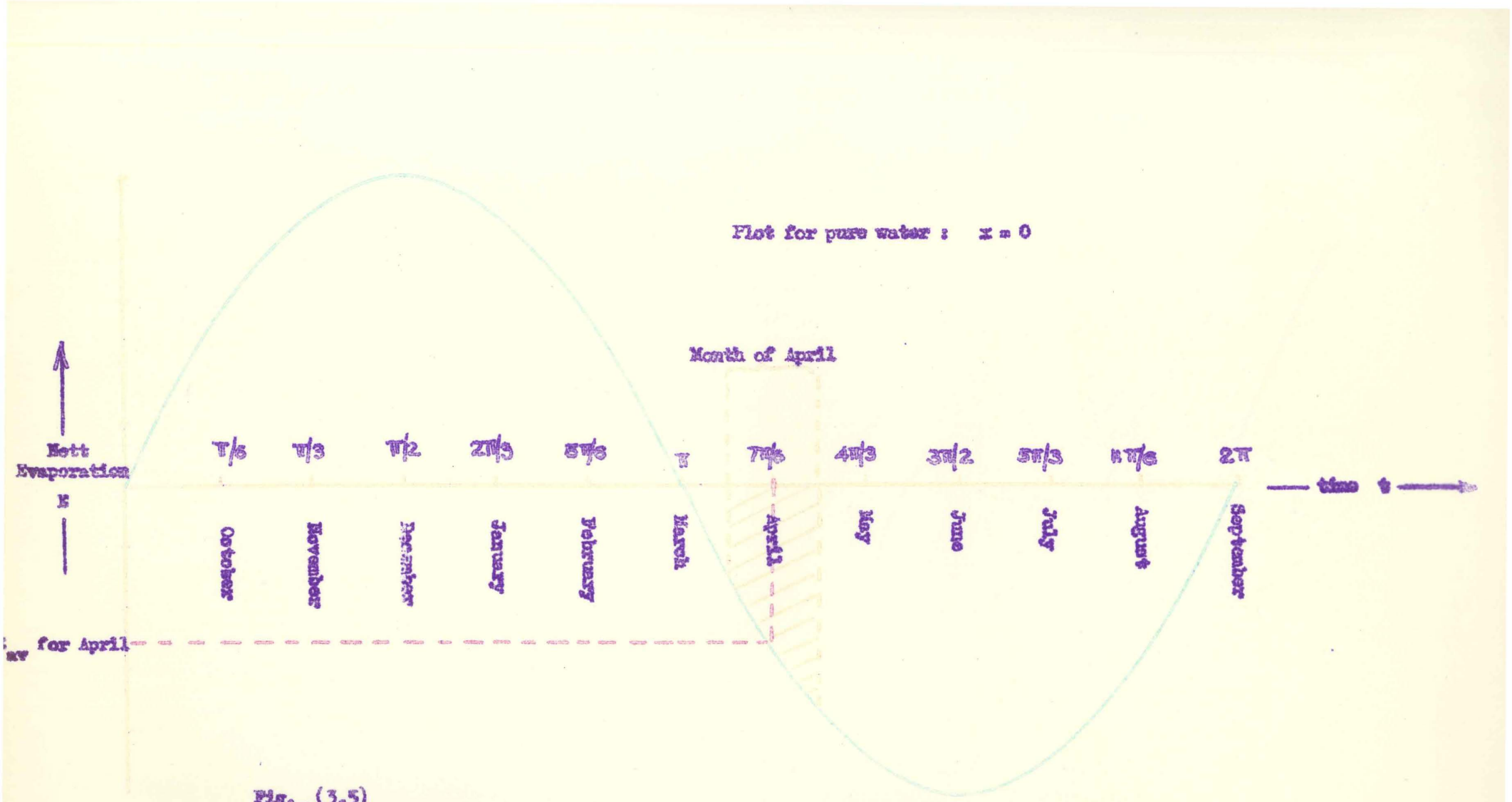


Fig. (3.5)

Variation of Evaporation Rate with time for pure water.

From data of table (3.4),

$q = 1.73$ inches per month

Table (3.4) gives annual average values of E ($=px + q$) for various values of x from which, on the average:

$p = -7.12$ inches per month

°Bé	x	Nett evap. coverage for (1) year $= px + q$	For $q = +1.73$ p
0	0	1.73	---
5	0.0258	1.56	-6.58
10	0.0545	1.38	-6.44
15	0.0810	1.17	-6.90
20	0.109	0.91	-7.51
25	0.137	0.61	-8.18

Average = -7.12

Table (3.4) - Values of p for various values of x

The variation in p is due to the divergence of the actual evaporation from that calculated by equation (3-27) since it does not follow the sinusoidal law exactly. The average value will be used.

The term (ax + b) is the amplitude of the sine curve given to the evaporation rate at concentration x.

When x = o, b is the amplitude of the pure water curve, so from previous:-

$b = +3.02$ inches/month

Also $\varepsilon = E - Eav$

$= (ax + b) = ax + 3.02$

This expression is tabulated in table (3.5) from which, on the average,

$a = -4.0$ inches/month

Hence, for Lake Grassmere, the nett evaporation rate E, is given by:-

$$E = (3.02-4.0x) \sin t + (1.73 - 7.12x) \quad \text{--- (3-29)}$$

	0°Be				5°Be				10°Be				15°Be				20°Be				25°Be			
MONTH	E		ε		E		ε		E		ε		E		ε		E		ε		E		ε	
JUNE	-1.26		2.99		-1.33		2.89		-1.44		2.82		-1.53		2.70		-1.65		2.56		-1.75		2.36	
DECEMBER	+4.77		3.04		+4.50		$2\overset{+}{.}94$		+4.24		$2\overset{+}{.}86$		+3.90		$2\overset{+}{.}73$		+3.50		$2\overset{+}{.}59$		2.95		$2\overset{+}{.}34$	
YEARS AV. "/month	+1.73		$3\overset{\pm}{.}02$		+1.56		$2\overset{\pm}{.}92$		+1.38		$2\overset{\pm}{.}84$		+1.17		$2\overset{\pm}{.}72$		0.91		$2\overset{\pm}{.}58$		0.61		$2\overset{\pm}{.}35$	
	x	ax + 3.02	ax	a	x	ax + 3.02	ax	a	x	ax + 3.02	ax	a	x	ax + 3.02	ax	a	x	ax + 3.02	ax	a	x	ax + 3.02	ax	a
	0	3.02	0	-	.0258	2.92	$0.\overline{10}$	$3.\overline{88}$	.0545	2.84	$0.\overline{18}$	$3.\overline{5}$	.081	2.72	$0.\overline{30}$	$3.\overline{71}$	.109	2.58	$.\overline{44}$	$4.\overline{04}$	.137	2.35	$.\overline{67}$	$4.\overline{88}$

Average value of a = $\frac{20.01}{5}$

i.e. a = $\underline{-4.0}$

Table (3.5) - Values of a for various values of x.

This expression for evaporation will have an uncertainty based on the doubtful nature of the data from which the constants a, b, p and q have been calculated, but will be used as a convenient expression. There appears no value in stating a probable error since the data for comparison with this expression is most unreliable and little meaning could be attached to such an error.

It should be noted that it is assumed that evaporation rate is a function of concentration and time only and is independent of the depth of brine. From what has been written above (see § 2) it is clear that the energy absorbed, energy losses and consequently the water evaporated, are functions of the brine depth but in absence of any quantitative information of the dependence of E upon the depth of brine, h, equation (3-29) will be assumed to hold for all h. Thus the constants a, b, p, and q in the evaporation formula, will be assumed independent of h.

§ 4.

Residence time:

For all calculations involving time, a reasonable time basis must be used which must be related to the residence time, t_r. That is, the time required under existing meteorological conditions for a unit of sea water to enter a pond system and leave it as saturated brine.

The minimum value that this residence time may have, is for evaporation taking place in a system containing an infinite number of ponds located say, in a finite area.

If this minimum residence time is much greater than one year, the variation of evaporation rate with time can be ignored and simple yearly averages used for each brine concentration.

That is, E = function of x only if the residence time t_r is $\gg$ 1 year. If, however, the residence time is of the order of 1 year, a time basis, which is small compared with the actual value of t_r, must be used and the variation of evaporation rate with time must be considered. That is: E = function of x and t.

The minimum value of t_r can be evaluated for an infinite number of ponds in a system. This is equivalent to having one pond of finite area in which streamline (piston-type) flow is taking place, i.e. where there is no back mixing.

Consider such a finite pond of length L, surface area A, and depth h. Assume the residence time t_r is very much greater than 1 year, so that E = function of x only, and average values of flow can be considered. If the calculated value of t_r based on these assumptions, is found to be much greater than 1 year, then these assumptions are valid. If t_r as calculated, is of the order of 1 year, then this value is in error, but E must then be taken as a function of x and t. In fact, it is this point which must be proved rather than a precise value of t_r.

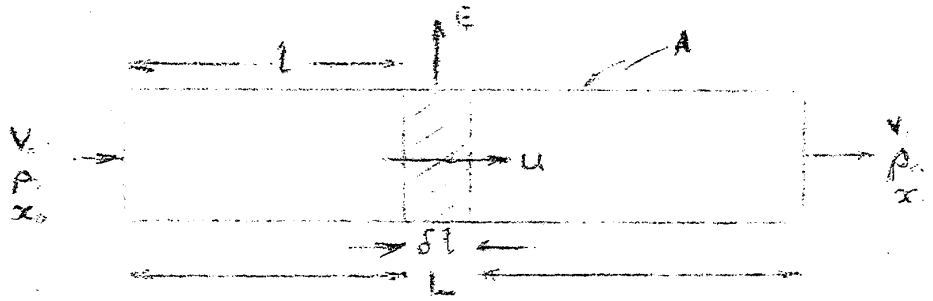


Fig. 4.1 The Residence time for an infinite number of ponds.

Let V be the mass flow rate of brine solution, V_o entering at sea water concentration, V_n leaving as saturated brine. See figure (4.1).

Consider an element l from the inlet of thickness δl and let the values of brine density, concentration, linear velocity and evaporation rate in that element be ρ, x, u, E respectively. Assuming no back mixing, a slice of fluid ρl thick enters the system, and due to evaporation from it, flows with decreasing velocity and decreasing thickness, towards the outlet.

The velocity at l = $u = \frac{V}{\rho} \frac{L}{Ah}$

$\therefore$ The time to traverse the element δl with constant velocity u

$$= \frac{\delta l}{u}$$

$\therefore$ The residence time for an infinite number of ponds in length L

$$= \int_o^{t_r} \frac{dl}{u}$$

$$= \int \frac{\rho}{V} \cdot \frac{Ah}{r} \cdot dl$$

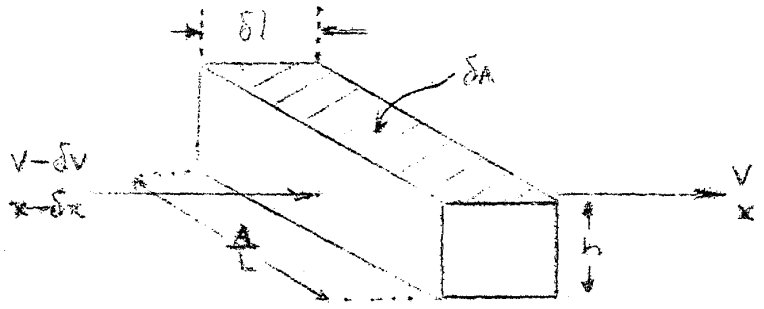
Further, as it is assumed that there is no variation of meteorological conditions with time, i.e. average conditions apply continuously, there will be no accumulation of salt at any point and so a material balance on the salt will give:

$$V_o x_o = Vx = V_n x_n$$

Hence the residence time -

$$= \frac{1}{V_o x_o} \frac{hA}{L} \int_o^L \rho x \, dl \qquad \text{---- (4-1)}$$

It is necessary to relate x to l



A material balance over the element δl, if the concentration changes by δx due to evaporation only, gives:

Weight of water evaporated in δl =(weight of solution entering the element) - (weight of solution leaving it)

in the limit as $\delta l \to o$

$$E.\frac{A}{L} dl = V_n x_n \left(\frac{1}{x} - \frac{1}{x+dx} \right)$$

$$\therefore \quad \frac{A}{L} dl = \frac{V_n x_n}{E} \left(\frac{1}{x} - \frac{1}{x + dx} \right)$$

$$= \frac{V_n x_n}{E x} - \frac{V_n x_n}{E} \left[\frac{1}{x\left(1 + \frac{dx}{x}\right)} \right]$$

$$= \frac{V_n x_n}{Ex} - \frac{V_n x_n}{Ex} \left(1 + \frac{dx}{x} \right)^{-1}$$

But dx is small of x and hence

$$\left(1 + \frac{dx}{x}\right)^{-1} = 1 - \frac{dx}{x} \quad \text{to the first order}$$

$$\text{i.e.} \quad \frac{A}{L} dl = \frac{V_n x_n}{x E} \frac{dx}{x}$$

and integrating between the limits of: (x= x_o when l =0
(x= x when l =1

gives: $l/\frac{L}{A} = V_n x_n \int_{x_o}^{x} \frac{dx}{x^2 E}$ ------(4 - 2)

The solution of equations (4 - 1) & (4 - 2) to give the residence time must be done graphically.

From the curve of E versus x, a plot of $\frac{1}{x^2 E}$ versus x can be made, and so given any output of saturated brine, V_n, the value of $l/\frac{L}{A}$ for any x can be calculated from the area under the curve defined by equation (4 - 2).

$$l = L \quad \text{when} \quad x = x_n$$

Figure (4.2) is such a plot giving values shown in table (4.1)

Such a procedure could give values of $l/\frac{L}{A} V_n x_n$ ($= l/\frac{L}{A} V_o x_o$) which could be used for the evaluation of equation (4 - 1) for the residence time.

viz: Residence time $t_r = h \int_o^L \rho x \, d\left(l\frac{L}{A} V_o x_o\right)$ ---- (4 - 3)

This residence time is independent of L, A, or the throughput, as of course it should be.

Also for equation (4 - 3)

$$\rho = \frac{\rho w}{1 - kx} = \frac{145 \rho w}{145 - {}^{o}Bé} \qquad ----(4 - 4)$$

$$\text{i.e.} \quad x = \frac{\rho - \rho w}{k\rho} \text{ --(4 - 5)}$$

${}^{o}Bé$ = density expressed in degrees Baume'

Where ρ_w = density of water = 62.4 lb/ft^3

k = $\frac{\text{Total weight of all salts in the brine}}{\text{Weight of NaCl in the brine}}$

For N.Z.: The percentage of NaCl in sea water = 2.787%

The percentage of all salts in sea water = 3.518%

$\therefore$ k for sea water = 1.262

J. Usiglio (Ref. 29) has given figures for the Mediterranean from which k can be calculated for various brine concentrations. The results appear in table 4.0. From these figures it appears 1.27 is a good average figure to use for k.

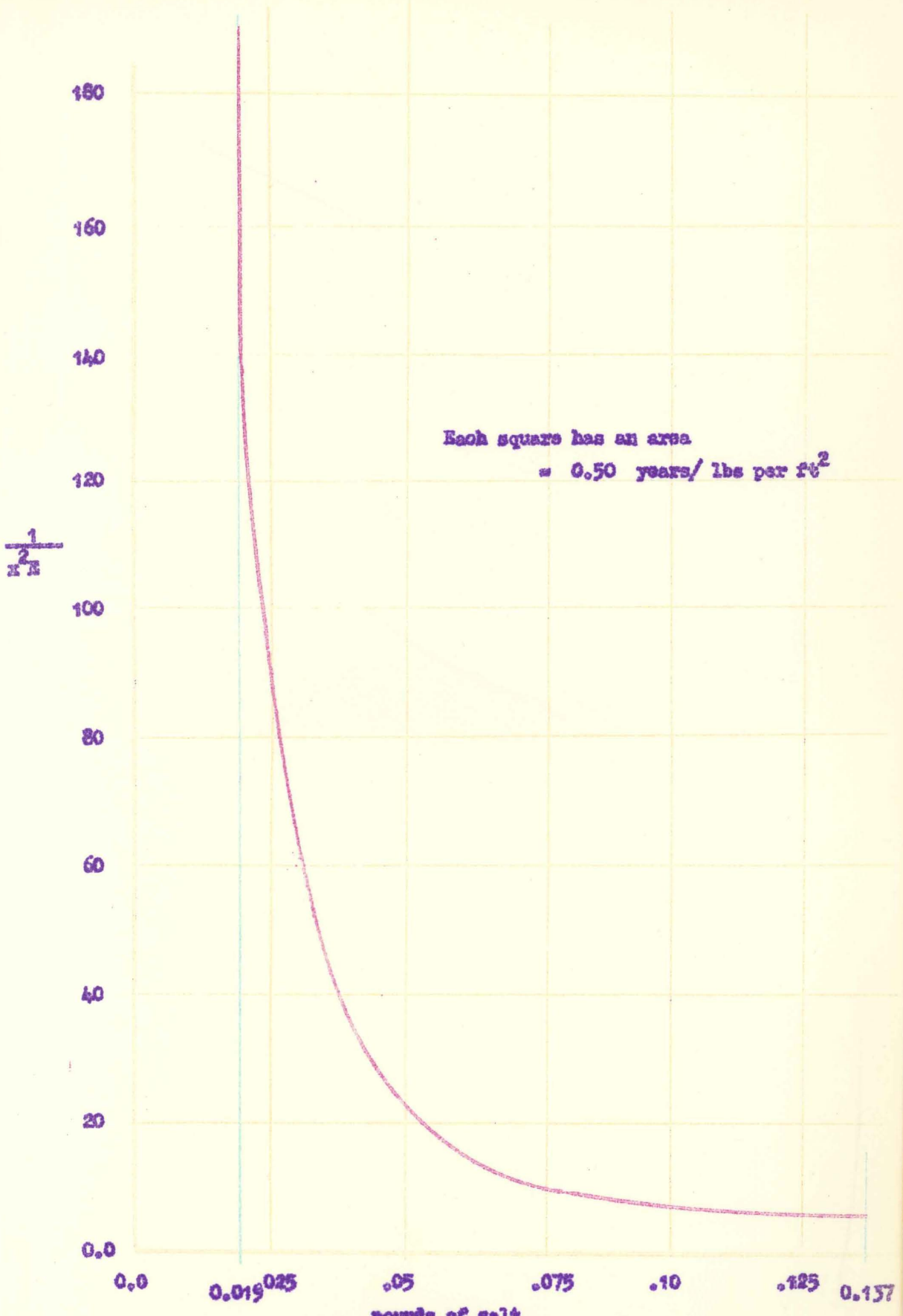


Fig. (4.2)

Evaluation of $1/I_A$

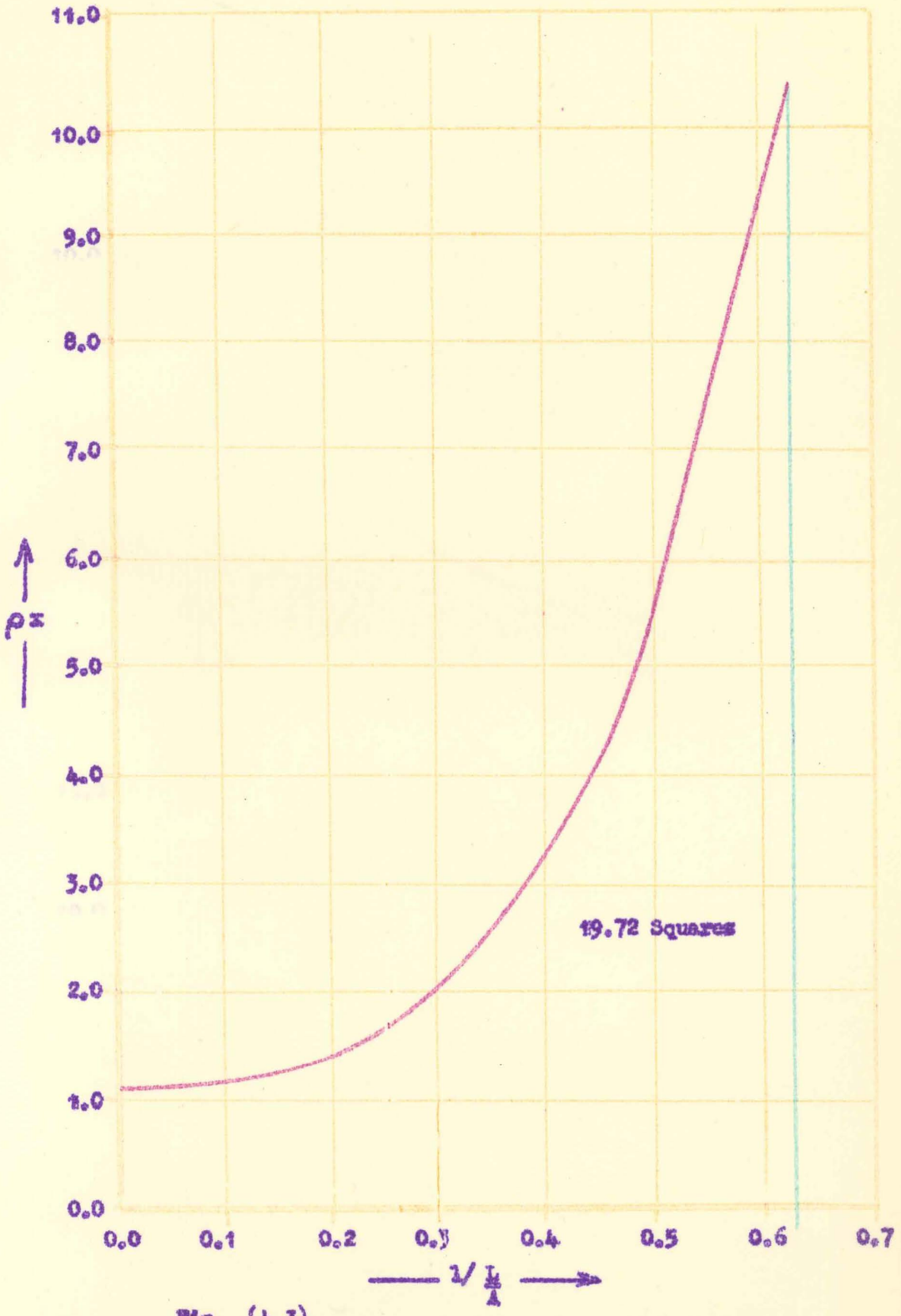


Fig. (4.3)

Determination of Residence Time

°Bé	SALTS DEPOSITED	SALTS STILL in SOLUTION.	NcCL STILL in SOLUTION.	k
3.5	0.000	38.4458	29.6894	1.295
6.9	0.0672	38.3786	"	1.290
11.1	0.0672	38.3786	"	1.290
13.6	0.0672	38.3786	"	1.290
16.3	0.6802	37.7656	"	1.27
20.6	1.2422	37.2036	"	1.255
24.1	1.5862	36.8596	"	1.236
25.3	4.9102	33.5356	26.4280	1.130

Table 4.0 - Values of the ratio

$$k = \frac{(\text{Weight of all salts in a sample of solution})}{(\text{Weight of NaCl in the same sample})}$$

This procedure gives the plot of Figure (4.3) from which, the residence time = 1.59 h years

^{o}Bé	From 4-4 lb/ft^3	From 4-5 x lb/lb	Nett E inches/year	$\frac{1}{x^2 E}$	$1/\frac{L}{A} V_o x_o$	ρx lb/ft^3
0.0	62.4	0.000	20.71	∞		
2.0	63.4	0.0125	20.0	320		
3.5	64.0	0.0199	19.35	132	0.000	1.275
5	64.5	0.0258	10.75	80	0.1161	1.665
10	67.0	0.0545	16.60	20.3	0.3371	3.650
15	69.5	0.0810	14.07	10.8	0.4089	5.63
20	72.3	0.1090	10.97	7.7	0.4573	7.89
25	74.5	0.1370	7.27	7.35	0.4960	10.34

Table 4.1 - Various values required for the evaluation of equations (4 - 2) and (4 - 3).

Thus, the residence time for an infinite number of ponds evaporating under average meteorological conditions at Lake Grassmere, (N.Z.) would be, to the first order, 1.6h years ---- (4 - 6)

By far the greatest error involved in this calculation is due to the uncertainty of E.

Now for a small time δt_r, from (4 - 3)

$$\delta t_r = h\rho x\, \delta\left(1/\frac{L}{A} V_o x_o\right)$$

$$= h\rho x \frac{\delta x}{x^2 E} \qquad \text{from (4 - 2)}$$

$$= \frac{h\rho\, \delta x}{x E} \qquad \text{----- (4 - 7)}$$

Thus applying the theory of probable errors (Ref. 43), if R_i is the probable error in the term i,

$$(R_{\delta t_r})^2 = \left[\frac{\partial(\delta t_r)}{\partial h}\right]^2 R_h^2 + \left[\frac{\partial(\delta t_r)}{\partial \rho}\right]^2 R_\rho^2 + \left[\frac{\partial(\delta t_r)}{\partial(\delta x)}\right]^2 R_{\delta x}^2$$

$$+ \left[\frac{\partial(\delta t_r)}{\partial x}\right]^2 R_x^2 + \left[\frac{\partial(\delta t_r)}{\partial E}\right]^2 R_E^2$$

But $\rho = f(x)$, and assuming h and x are not in error

$$R_{\delta t_r} = \pm \left[\frac{\partial (\delta t_r)}{\partial E} \right]_{x,h} \cdot R_E$$

From equation (4 - 7), since x is constant

$$\frac{\partial (\delta t_r)}{\partial E} = - \frac{h\rho \cdot \delta x}{xE^2}$$

$$\therefore \quad R_{\delta t_r} = \mp \frac{h \rho \delta x}{xE^2} \cdot R_E$$

but $$R_{t_r} = \Sigma R_{\delta t_r}$$

i.e. $$R_{t_r} = R_E \pm \int_{x_o}^{x_n} \frac{h\rho \, dx}{xE^2}$$

So for average values of h, ρ, x, E

$$R_{t_r} = \frac{h \rho}{xE^2} \Delta x. R_E \quad \text{---- (4-8)}$$

where Δx = range of x in this problem.

Take average S.G. = 1.2 (Ref. 29) x = 0.0199, x_n = 0.137,

Δx = 0.1 (say), x (average) = 0.07, however the evaporation rate is greater at low concentrations, a time biased average is = 12 inches/year = 62.4 lb/ft. /year.

From S (3) R_E = of $\pm$ 15% i.e. $\pm$ 1.8 inches/year

when E = 12 inches/year.

Hence $$R_{t_r} \pm \frac{h \times 1.2 \times 62.4}{0.1 \times (62.4)^2} \times 0.1 \times \frac{1.8}{12} \times 62.4$$

$$\pm \underline{0.18 \text{ h years}} \quad \text{(i.e. 12.5\%)}$$

$$\therefore \quad t_r \quad \underline{1.6h \pm 0.18h \text{ years}} \quad \text{---- (4-9)}$$

For h = 1 foot depth

$$t_r = 1.6 \pm 0.2 \text{ years.}$$

This means that the residence time can not be considered large compared with 1 year, so that the rate of evaporation from any one "block" of brine, depends on the time as well as the concentration of the brine.

Certainly, $V_n x_n \neq V x \neq V_o x_o$ and there may be a considerable accumulation of salt in the system in any finite time.

A time basis of one month was chosen, in which evaporation rate varied smoothly with time, from the beginning to the end of the month, in a way defined by a sine curve of period $2\pi \equiv 1$ year.

No finer division of time could be used since evaporation would then have an uncertainty of at least 100%, clearly indicating the need for accurate evaporation versus concentration data for small time intervals.

If $V_n x_n$ was evaluated, given that W tons of salt per year were required to be harvested and that a fraction g of the salt in the brine was sold,

$$V_n x_n = \frac{W}{g} \text{ tons/year}$$

J. Usiglio (Ref. 29), gives figures from which g could be calculated roughly.

Let saturated brine $= 25^\circ$ Bé'

Let the bittern be discarded at at 29.6° Bé'

Then the salt NaCl deposited $= \frac{23.42}{23.42 + 3.67} \times 100\%$

$= 86.5\%$ of that present.

If 60% of the deposited salt is harvested, washed and sold, then

$$g = \frac{86.5}{100} \times \frac{6}{10} \times 100\%$$

$$= 51.9\%$$

§ 5.

Material Balances:

On the basis of this variation of evaporation rate with concentration and time, material balances across a system of ponds were investigated to ascertain the output of saturated brine from the system.

Consider a system of n ponds, 1 - n, with sea water flowing into pond 1 at a rate F_o (volume of brine flowing per unit time) and saturated brine (say 25° Bé) flowing from pond n at the rate F_n.

Let E_1, E_2, E_3 ----E_n be the instantaneous evaporation rates (Volume of water evaporated from the pond per unit area of pond surface per unit time);

A_1, A_2, A_3 ----A_n be the surface areas of the ponds;

h_1, h_2, h_3 ----h_n be the depths of brine;

ρ_1, ρ_2, ρ_3 ----ρ_n be the densities of brine;

x_1, x_2, x_3 ---x_n be the ratio of the weight of salt NaCl per unit weight of brine solution; k_1, k_2, k_3 --- k_n be the ratio of the weight of all salts in the brine to unit weight of NaCl in the brine; and let F_1, F_2 --- F_n be the flows of brine from each of the ponds respectively. See Figure 5.1

x is related to density ρ for, if the specific gravity of the brine in pond r be

$$S_r = \frac{\text{weight of unit volume of the brine}}{\text{weight of same ---------- water}}$$

i.e. $\dfrac{\rho_r}{\rho_w} = S_r$ where ρ_w = density of water

Also $x_r = \dfrac{\text{weight of NaCl in the brine sample from pond } r}{\text{weight of the brine sample from pond } r}$

$$\therefore \quad k_r x_r = \frac{\text{weight of all salts in the brine sample for pond } r}{\text{weight of the brine sample}}$$

$$\therefore \quad 1 - k_r x_r = \frac{\text{weight of water in the brine sample}}{\text{weight of the brine sample}}$$

So, assuming the law of additive volumes (Ref. 3)

$$\frac{\rho_r}{\rho_w} = S_r = \frac{1}{1 - k_r x_r}$$

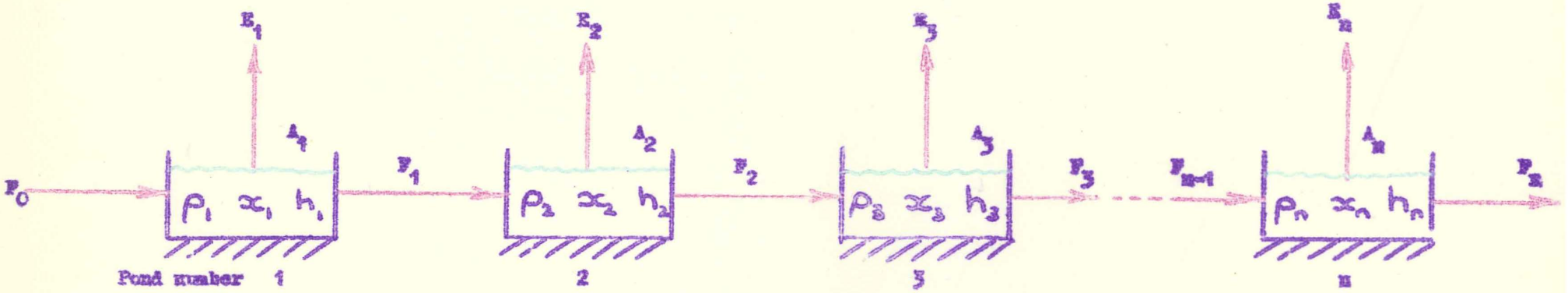


Fig. (5.1)

Schematic diagram of the layout of the ponds in a system of n ponds of total area A_t producing saturated brine from sea water.

$$\text{i.e.} \quad \rho_r = \frac{\rho_w}{1 - k_r x_r} \qquad \text{---- (5 - 1)}$$

$$\text{or} \quad x_r = \frac{\rho_r - \rho_w}{k_r \rho_r} \qquad \text{---- (5 - 2)}$$

Considering such a system as defined above, it is possible to write down instantaneous material balances.

Assume that no losses of brine occur. That is, there is no seepage of brine out or ground water into the pond system. This is quite a reasonable assumption, particularly after a few years operation when calcium sulphate has been deposited in the ponds of concentration about 15 to 23 ^{o}Bé and salt crystals have sealed the crystalliser. A good selection of site will, of course, mean impermeable mud bottoms.

Also assume that the ponds are completely mixed so that the brine leaving the pond is of the same concentration as the brine in the pond. This is a reasonable assumption, particularly where wind velocity is high.

A recent paper by Danckwerts (Ref. 75) discusses the estimation of distribution functions for continuous flow systems in which neither piston flow or complete mixing takes place. A case of large ponds treated by models is referred to. With such distribution functions any variation from complete mixing in the ponds could be allowed for.

In an element of time t, considering instantaneous values;

A volume balance

$$F_o = F_1 + E_1 A_1 + \frac{dv}{dt}$$

$$= F_1 + E_1 A_1 + \frac{dh_1}{dt} \qquad \text{---- (5 - 3a)}$$

Similarly for ponds 2, 3, ---- n

$$F_1 = F_2 + E_2 A_2 + A_2 \frac{dh_2}{dt} \qquad \text{---- (5 - 3b)}$$

$$F_2 = F_3 + E_3 A_3 + A_3 \frac{dh_3}{dt} \qquad \text{---- (5 - 3c)}$$

$$F_{n-2} = F_{n-1} + E_{n-1} A_{n-1} + A_{n-1} \frac{dh_{n-1}}{dt} \qquad \text{---- (5 - 3d)}$$

$$F_{n-1} = F_n + E_n A_n + A_n \frac{dh_n}{dt} \qquad \text{---- (5 - 3e)}$$

If it is postulated that the depth of brine in a pond be maintained constant, then $\frac{dh}{dt} = 0$ and generally for pond r,

$$F_{r-1} = F_r + E_r A_r \quad \text{---- (5 - 4)}$$

Overall weight balance

In any period of time an accumulation can take place in any pond so that at any instant there will be rate of accumulation $\frac{dw}{dt}$ and hence the weight in to a pond = weight out from the pond + accumulation of weight in the pond.

For pond 1 -

$$F_o \rho_o = F_1 \rho_1 + E_1 A_1 \rho_w + \frac{dw_1}{dt}$$

$$= F_1 \rho_1 + E_1 A_1 \rho_w + A_1 \frac{d(h_1 \rho_1)}{dt} \quad \text{---- (5 - 5a)}$$

Similarly for ponds 2, 3, ---------- n.

$$F_1 \rho_1 = F_2 \rho_2 + E_2 A_2 \rho_w + A_2 \frac{d(h_2 \rho_2)}{dt} \quad \text{---- (5 - 5b)}$$

$$F_2 \rho_2 = F_3 \rho_3 + E_3 A_3 \rho_w + A_3 \frac{d(h_3 \rho_3)}{dt} \quad \text{---- (5 - 5c)}$$

etc to -

$$F_{n-1} \rho_{n-1} = F_n \rho_n + E_n A_n \rho_w + A_n \frac{d(h_n \rho_n)}{dt} \quad \text{---- (5 - 5n)}$$

or generally for pond r -

$$F_{r-1} \rho_{r-1} = F_r \rho_r + E_r A_r \rho_w + A_r \frac{d(h_r \rho_r)}{dt} \quad \text{---- (5 - 6)}$$

$$= F_r \rho_r + E_r A_r \rho_w + A_r \rho_r \frac{dh_r}{dt} + A_r h_r \frac{d\rho_r}{dt}$$

and again for $\frac{dh}{dt} = 0$

$$F_{r-1} \rho_{r-1} = F_r \rho_r + E_r A_r \rho_w + A_r h_r \frac{d\rho_r}{dt} \quad \text{---- (5 - 7)}$$

Salt Balance

Again for any time interval, Weight of salt entering a pond = salt leaving the pond + accumulation of salt, so that instantaneously;

For pond 1 -

$$F_o \rho_o x_o = F_1 \rho_1 x_1 + \frac{dw_{Nacl}}{dt}$$

$$= F_1 \rho_1 x_1 + A_1 \frac{d(h_1 \rho_1 x_1)}{dt} \quad \text{---- (5 - 8a)}$$

And similarly -

$$F_1 \rho_1 x_1 = F_2 \rho_2 x_2 + A_2 \frac{d(h_2 \rho_2 x_2)}{dt} \quad \text{---- (5 - 8h)}$$

etc to -

$$F_{n-1} \rho_{n-1} x_{n-1} = F_n \rho_n x_n + A_n \frac{d(h_n \rho_n x_n)}{dt} \quad \text{---- (5 - 8n)}$$

or generally for pond r -

$$F_{r-1} \rho_{r-1} x_{r-1} = F_r \rho_r x_r + A_r \frac{d(h_r \rho_r x_r)}{dt} \quad \text{---- (5 - 9)}$$

$$= F_r \rho_r x_r + A_r h_r \frac{d(\rho_r x_r)}{dt} + A_r \rho_r x_r \frac{d(h_r)}{dt}$$

And again for $\frac{dh}{dt} = 0$

$$F_{r-1} \rho_{r-1} x_{r-1} = F_r \rho_r x_r + A_r h_r \frac{d(\rho_r x_r)}{dt} \quad \text{---- (5 - 10)}$$

Water Balance

It is clear that a water balance for pond r could be written from equations (5-7) and (5-10) to give when $\frac{dh}{dt} = 0$

$$F_{r-1} \rho_{r-1} (1-x_{r-1}) = F_r \rho_r (1-x_r) + A_r h_r \frac{d \rho_r (1-x_r)}{dt} \quad \text{--- (5 - 11)}$$

§ 6.

Yields of Saturated Brine:

A direct use of the material balances derived in § 5 is to investigate the output of saturated brine from a system of ponds. Such an investigation can then be used as a basis for the determination of several important design factors.

(i) The best output of saturated brine to be expected from a given total area of ponds in an average year,

(ii) The best number of ponds into which the given area should be subdivided to give this best output,

(iii) The best size of the ponds both actual size and size relative to each other,

(iv) How the depth of brine in any pond affects the output of the total system.

It should be noted that in (i) and (ii), the best output from a given area must be when that area is subdivided into an infinite number of ponds, or alternatively when there is no subdivision of the area and operation takes place in one pond with streamlined flow. It is also clear that one pond, perfectly mixed, will give the poorest yield of saturated brine and the output will increase as the subdivision increases. (See Appendix) Thus parts (i) and (ii) should give a plot of output of saturated brine from a system of ponds against the number of ponds in the system and time. Economics would then decide the optimum number of ponds to give the greatest output of saturated brine per unit cost.
In (iii) the relation of the pond areas to each other, and the actual sizes of ponds, must be related in practice to the lie of the land. Only a detailed investigation would show the theoretically best lay out.

The results of brine depth considerations on output should be related to the investigation suggested in § 2 for optimum depth.

The material balances in § 5 only apply instantaneously. It is clear that integration over a finite time interval in which flow rate, concentration of brine and evaporation rates all vary, will give the change in flow produced by changing conditions

E is expressible in terms 0 x & t, see § 3, by -

$$E = (ax_r + b) \sin t + (px_r + q) \qquad \text{—— (3 - 27)}$$

Whence $\left(\frac{\partial E}{\partial t}\right)_{x_r} = (ax_r + b) \cos t$

$$\frac{(\partial E_r)}{(\partial x_r)_t} = a \sin t + p$$

$$\frac{dE_r}{dt} = \left(\frac{\partial E}{\partial t}\right)_x + \left(\frac{\partial E}{\partial x}\right)_t \frac{dx}{dt} \qquad \text{—— (3 - 28)}$$

On this basis, flow can be expressed in a series of differential equations, a solution to which would yield flow as a function of time only.

§ 6/I General Case: For n ponds.

From (5 - 4)

$$F_{r-1} = F_r + E_r A_r$$

If over a time interval δt, x changes by δx, F by δF etc, then -

$$\left(\frac{\partial F_{r-1}}{\partial t}\right)_x \delta t + \left(\frac{\partial F_{r-1}}{\partial x}\right)_t \delta x_{r-1} = \left(\frac{\partial F_r}{\partial t}\right)_x \delta t + \left(\frac{\partial F_r}{\partial x}\right)_t \delta x_r$$

$$+ A_r \left(\frac{\partial E_1}{\partial t}\right)_x \delta t + \left(\frac{\partial E}{\partial x}\right)_t \delta x_r \qquad \text{——(6-1)}$$

Equation (6 - 1) applies to all ponds and summing over n ponds gives -

$$\left(\frac{\partial F_o}{\partial t}\right)_x \delta t + \left(\frac{\partial F_o}{\partial x}\right)_t \delta x_o = \left(\frac{\partial F_n}{\partial t}\right)_x \delta t + \left(\frac{\partial F_n}{\partial x}\right)_t \delta x_n$$

$$+ \sum_{r=1}^{r=n} A_r \left(\frac{\partial E}{\partial t}\right)_x \delta t + \left(\frac{\partial E}{\partial x}\right)_t \delta x$$

or in the limit as $\delta t \to 0$

$$\left(\frac{\partial F_o}{\partial t}\right)_x + \left(\frac{\partial F_o}{\partial x}\right)_t \frac{dx_o}{dt} = \left(\frac{\partial F_n}{\partial t}\right)_x + \left(\frac{\partial F_n}{\partial x}\right)_t dx_n/dt \sum_{r=1}^{n} A_r \frac{(dE)}{dt_r} \quad \text{--- (6-2)}$$

where $\frac{dE_r}{dt} = \left(\frac{\partial E_r}{\partial t}\right)_x + \left(\frac{\partial E}{\partial x}\right)_t \frac{dx}{dt}$

However, to produce saturated brine continually $x_n = x_{sat}$ and $\frac{dx_n}{dt} = 0$

<u>Also</u> x_o = sea water concentration. <u>So</u> $\frac{dx_o}{dt} = 0$

$$\therefore \left(\frac{\partial F_o}{\partial t}\right)_x - \left(\frac{\partial F_n}{\partial t}\right)_x = \sum_{r=1}^{n} A_r \frac{(dE_r)}{dt} \quad \text{---- (6-3)}$$

Further, from (5 - 10), instantaneously -

$$F_{r-1} \rho_{r-1} x_{r-1} = F_r \rho_r x_r + A_r h_r \frac{d(\rho_r x_r)}{dt}$$

so variation over a time interval dt gives in the limit, as $\delta t \to 0$,

when summation is carried out over n ponds:

and given $\frac{dx_o}{dt} = 0$, $\frac{dx_n}{dt} = 0$:

$$\left(\frac{\partial F_o}{\partial t}\right)_x \rho_o x_o - \left(\frac{\partial F_n}{\partial t}\right)_x \rho_n x_n = \frac{\partial}{\partial t} \sum_{r=1}^{n} h_r A_r \frac{d(\rho_r x_r)}{dt} \quad \text{---- (6 - 4)}$$

The instantaneous overall weight balance is given generally in (5-7) as -

$$F_{r-1}\rho_{r-1} = F_r \rho_r + E_r A_r \rho_w + A_r h_r \frac{d\rho_r}{dt}$$

Summation for the n ponds and differentiation with respect to time gives:

$$\left(\frac{\rho F_o}{\partial t}\right)_x \rho_o - \left(\frac{\rho F_n}{\partial t}\right)_x \rho_n = \rho \sum_{r=1}^{n} A_r \frac{dE}{dt} + \sum_{r=1}^{n} h_r A_r \frac{d^2\rho_r}{dt^2} \quad \text{---- (6 - 5)}$$

To solve for the variation of output of saturated brine $\left(\frac{\partial F_o}{\partial t}\right)_x$, $\left(\frac{dx}{dt}\right)$;

$\frac{d^2\rho}{dt^2}$, $\frac{d(\rho_r x_r)}{dt}$, must be eliminated from equations (6-3), (6-4), (6-5) with the derivatives of (5-1)

i.e. $$\rho_r = \frac{\rho_w}{1 - k_r x_r}$$

$$\therefore \frac{d\rho_r}{dt_r} = \frac{\rho_w k_r}{(1-k_r x_r)^2} \frac{dx}{dt} \quad \text{---- (6-6)}$$

and $$\frac{d^2\rho_r}{dt^2} = \frac{2\rho_w k_r^2}{(1-k_r x_r)^3} \left(\frac{dx_r}{dt}\right)^2 + \frac{\rho_w k_r}{(1-k_r x_r)^2} \frac{d^2x_r}{dt^2} \quad \text{--- (6-7)}$$

and $$\frac{d(\rho_r x_r)}{dt} = \frac{\rho_w}{(1-k_r x_r)^2} \frac{dx_r}{dt} \quad \text{---- (6 - 8)}$$

and $$\frac{\partial}{\partial t} \frac{d(\rho_r x_r)}{dt} = \frac{\rho_w}{(1-k_r x_r)^2} \frac{d^2x_r}{dt^2} \quad \text{---- (6 - 9)}$$

In addition to these equations, an expression of x as a function of t will be required to enable an integration over a time period. From (6-3)

$$\left(\frac{\partial F_o}{\partial t}\right)_x - \left(\frac{\partial F_n}{\partial t}\right)_x = \sum_{r=1}^{n} A_r \left(\frac{\partial E_r}{\partial t}\right)_x + \sum_{r=1}^{n} A_r \left(\frac{\partial E}{\partial x}\right)_t \frac{dx_r}{dt}$$

but from (3 - 27)

$$\left(\frac{\partial E_r}{\partial t}\right)_x = (ax_r + b)\ \cos t$$

$$\left(\frac{\partial E_r}{\partial x}\right)_t = a \sin t + p$$

Thus -

$$\left(\frac{\partial F_o}{\partial t}\right)_x - \left(\frac{\partial F_n}{\partial t}\right)_x = \sum_{r=1}^{n} A_r (ax_r + b) \cos t + \sum_{r=1}^{n} A_r (a \sin t + p) \frac{dx_r}{dt}$$

$$= a \cos t \sum_{r=1}^{n} A_r x_r + n b \cos t + (a \sin t + p) \sum_{r=1}^{n} A_r \frac{dx_r}{dt}$$

$$\therefore \sum_{r=1}^{n} A_r \frac{dx_r}{dt} = \frac{\left(\frac{\partial F_o}{\partial t}\right)_x - \left(\frac{\partial F_n}{\partial t}\right)_x - n b \cos t - a \cos t \sum_{1}^{n} A_r x_r}{(a \sin t + p)} \quad \text{--(6-10)}$$

It is clear that substitution of (6-10) into (6-4), or (6-5) gives expression of undue complexity. For example substituting into (6-5)

$$\left(\frac{\partial F_o}{\partial t}\right)_x \rho_o - \left(\frac{\partial F_n}{\partial t}\right)_x \rho_n = \sum_{r=1}^{n} \left[\rho_w A_r \frac{dE}{dt} + h_r A_r \frac{d^2 \rho_r}{dt^2} \right]$$

$$= \rho_w \left[\left(\frac{\partial F_o}{\partial t}\right)_x - \left(\frac{\partial F_n}{\partial t}\right)_x \right] + \Sigma h_r A_r \frac{d^2 \rho_r}{dt^2}$$

From equation (6 - 3)

$$\text{i.e.} \left(\frac{\partial F_o}{\partial t}\right)_x (\rho_o - \rho_w) - \left(\frac{\partial F_n}{\partial t}\right)_x (\rho_n - \rho_x) = \sum_{r=1}^{n} \left[h_r A_r \frac{2 \rho_w k_r^2}{(1-k_r x_r)^3} \left(\frac{dx_r}{dt}\right)^2 + \frac{\rho_w k_r}{(1-k_r x_r)^2} \frac{d^2 x_r}{dt^2} \right]$$

From equation (6 - 7)

This expression could only be evaluated for $\frac{\partial F_n}{\partial t}$, for integration for ΔF_n, if the concentration of every pond was known at each instant and as a function of time; if $\frac{(dx_r)^2}{(dt)}$, $\frac{dx_r}{dt}$ could be expressed for each pond, and if $\frac{\left(\frac{\partial F_o}{\partial t}\right)}{\left(\frac{\partial t}{\partial t}\right)_x}$ could be eliminated.

Generally: If kx is small compared to 1

$$(1-kx)^{-2} = 1 + 2kx + \frac{(-2x-2-1)}{L^2} (-kx)^2 + \text{---}$$

In this problem x varies from 0.0199 (at sea water) to 0.137 (at 25° Bé' brine) and k from 1.28 down to 0.19 at the same concentrations. Thus kx varies from 0.024 to 0.027 so that kx is always very small and varies very little.

Also assume that the areas of all ponds are equal and that the depth of brine, h_r, in each pond not only remains constant in that pond but is constant throughout the system.

That is - $A_r = A$ for all r

$$h_r = h \text{ for all } r \text{ and all } t.$$

Also from table (4.0) of k_r versus x it is clear that up to 25° Bé', k_r remains practically constant and could be taken as 1.27 on the average (k_{av})

Thus $\frac{1}{(1-k_r x_r)^2}$ could be replaced by $1 + 2\,(kx)_{av}$.

and similarly $\frac{1}{(1-k_r x_r)^3}$ by $1 + 3\,(kx)_{av}$.

Thus -

$$\left(\frac{\partial F_o}{\partial t}\right)_x (\rho_o - \rho_w) \left(\frac{\partial F_n}{\partial t}\right)_x (\rho_n - \rho_w)$$

$$= hA\,2\rho_w \left[1 + 3\,(kx)_{av}\right] k_{av}^2 \sum_{r=1}^{n} \left(\frac{dx}{dt}\right)^2 + hA\rho_w\, k_{av} \left[1 + 2\,(kx)_{av}\right] \sum_{r=1}^{n} \frac{d^2x_r}{dt^2}$$

---- (6 - 11)

$\sum_{r=1}^{n} \frac{d^2x_r}{dt^2}$ can be replaced by differentiating equation (6-10): viz for all areas equal:

$$A \sum \frac{d^2x_r}{dt^2} = \left[(a \sin t + p)\left[\frac{(\partial^2 F_o)}{(\partial t^2)_x} - \frac{(\partial^2 F_n)}{(\partial t^2)_x}\right] + nb \sin t. \right.$$

$$\left. - aA \cos t \sum_{r=1}^{n} \frac{dx_r}{dt} + aA \sin t \sum_{r=1}^{n} x_r \right]$$

$$- \left[\left[\frac{(\partial F_o)}{(\partial t)_x} - \frac{(\partial F_n)}{(\partial t)_x} - n\,b \cos t - aA \cos t \sum_{r=1}^{n} x_r\right] \right.$$

Cont'd

$$\times \frac{1}{(a \sin t + p)^2} \qquad \text{---- (6 - 12)}$$

And substituting in for $\sum_{r=1}^{n} \frac{dx_r}{dt}$ produces a hopelessly complex statement.

Further for -

$$\sum_{r=1}^{n} \frac{(dx_r)^2}{(dt)} \equiv \left[\sum_{r=1}^{n} \frac{(dx_r)}{dt} \right]^2 - 2 \sum_{r=1}^{n-1} \left[\frac{dx_r}{dt} \cdot \frac{dx_{r-1}}{dt} \right]$$

it can be shown -

$$\sum_{r=1}^{n} \frac{(dx_r)^2}{(dt)} = \frac{\sin^2 t}{(a \sin t + p)^2} \sum_{r=1}^{n-1} (ax_r + b) - \frac{1}{A^2(a \sin t + p)^2} \sum_{r=1}^{n} \frac{(dF_r)^2}{dt}$$

$$- \frac{1}{A^2(a \sin t + p)^2} \left[\frac{(\partial F_o)}{(\partial t)_x} \frac{(\partial F_n)}{(\partial t)_x} \right] + \frac{1}{A^2(a \sin t + p)^2} \sum_{r=1}^{n-1} \frac{(\partial F_r)}{(\partial t)_x} \frac{(\partial F_{r+1})}{(\partial t)_x}$$

------(6 - 13)

Substitution of (6-12) and (6-13) into (6-11) and subsequent elimination of $\frac{(\partial F_o)}{(\partial t)_x}$ by simultaneous solution with equation (6-4) in which equation (6-9) viz: $\frac{\partial}{\partial t} \frac{d(\rho_r x_r)}{dt} = \frac{\rho_w}{(1-k_r x_r)^2} \frac{d^2 x_r}{dt^2}$ and equation (6-12) for $\frac{d^2 x_r}{dt^2}$ had been substituted, yields an expression for $\frac{(\partial F_n)}{(\partial t)_x}$ in terms of x, t, $\frac{(\partial F_r)}{(\partial t)_x}$ for values of r from 0 - n and knowns.

It is clear that this expression is insoluble for not only are all the $\frac{(\partial F_r)}{(\partial t)_x}$ unknown but x is an unknown function of t for all the ponds. Further the complexity of the resulting expression would make it highly unsatisfactory for numerical evaluation.

In forming these equations, the summations over n ponds

have eliminated all the intermediate expressions which now appear necessary so that the general case remains insoluble. That is, the fundamental summation equations involving x_r, h_r, A_r, E_r, ρ_r become inadequate to solve the overall problem and equations relating x_{r-1} and x_{r+1} to x_r are also required

§ 6/2

Stepwise Procedure:

From equation (5 -8a)

$$F_o\rho_o x_o = F_1\rho_1 x_1 + A_1 \frac{d(h_1\rho_1 x_1)}{dt}$$

$$= F_1\rho_1 x_1 + A_1 h_1 \rho_1 \frac{dx_1}{dt} + A_1 x_1 \frac{d(h_1\rho_1)}{dt}$$

but from equations (5-5a)

$$A_1 \frac{d(h_1\rho_1)}{dt} = F_o\rho_o - F_1\rho_1 - E_1 A_1 \rho_w$$

$$\therefore F_o\rho_o x_o = F_1\rho_1 x_1 + A_1 h_1 \rho_1 \frac{dx_1}{dt} + F_o\rho_o x_1 - F_1\rho_1 x_1 - E_1 A_1 \rho_w x_1$$

$$\text{i.e. } F_o\rho_o(x_o - x_1) = A_1 h_1 \rho_1 \frac{dx_1}{dt} - E_1 A_1 \rho_w x_1$$

$$\text{or } \frac{dx_1}{dt} = \frac{E_1 x_1}{h_1} \frac{\rho_w}{\rho_1} - \frac{F_o \rho_o}{A_1 h_1 \rho_1} \cdot (x_1 - x_o) \quad \text{----(6-14)}$$

or by substituting from equation (5 - 1)

$$\frac{dx_1}{dt} = \frac{E_1 x_1}{h_1} (1 - k_1 x_1) - \frac{F_o}{A_1 h_1} \frac{1 - k_1 x_1}{1 - k_o x_o} \cdot (x_1 - x_o) \quad \text{----(6 - 15)}$$

For the second pond, a similar approach yields -

$$\frac{dx_2}{dt} = \frac{E_2 x_2}{h_2} (1 - k_2 x_2) - \frac{F_1}{A_2 h_2} \cdot \frac{1 - k_2 x_2}{1 - k_1 x_1} \cdot (x_2 - x_1) \text{ ----(6 - 16)}$$

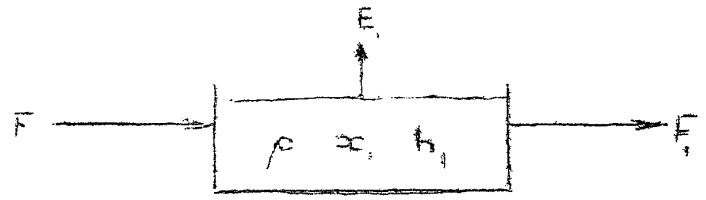
and generally for pond r -

$$\frac{dx_r}{dt} = \frac{E_r x_r}{h_r} (1 - k_r x_r) - \frac{F_{r-1}}{A_r h_r} \frac{1 - k_r x_r}{1 - k_{r-1} x_{r-1}} \cdot (x_r - x_{r-1}) \text{ --(6-17)}$$

These equations form the basis of a stepwise procedure.

Case I - A system of One Pond with perfect mixing:

The accompanying figure illustrates such a system.



To produce only saturated brine in the flow F_1, $\rho_1 = \rho$ saturated

$x_1 = x$ saturated.

It is assumed that the concentration of brine in the pond is the same as the brine leaving in flow F_1 and at no time must the concentration in the pond be anything but x_{sat}.

$$\text{i.e. } \frac{dx_1}{dt} = 0 \text{ for all } t$$

Thus from (6-17) for the case of one pond only:

$$\frac{E_1 x_1}{h_1}(1 - k_1 x_1) = \frac{F_o}{A_1 h_1} \cdot \frac{1 - k_1 x_1}{1 - k_o x_o}(x_1 - x_o)$$

and since $(1 - k_1 x_1) \neq 0$

$$\frac{E_1 A_1 x_1 (1 - k_o x_o)}{(x_1 - x_o)} = F_o$$

and substituting for E_1 from equation (3 - 27)

$$F_o = \left[(ax_1 + b)\sin t + (px_1 + q)\right]\frac{A_1 x_1}{(x_1 - x_o)}(1 - k_o x_o)$$

but $x_1 = x_{sat}$ so:

$$F_o = \left[(ax_{sat} + b)\sin t + (px_{sat} + q)\right]\frac{A_1 x_{sat}}{x_{sat} - x_o}(1 - k_o x_o) \quad \text{--- (6-18)}$$

which relates F_o to t only for the case of system with one pond.

For the output of saturated brine, F_1 -

$$F_o = F_1 + E_1 A_1 \text{ for constant } h_1 \quad (5 - 4)$$

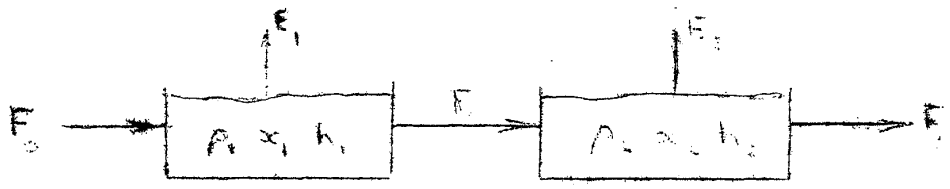
$$\text{i.e. } F_1 = F_o - E_1 A_1$$

$$\therefore \frac{F_1}{A_1} = \frac{F_o}{A_1} - E_1$$

$$= \left[(ax_1 + b) \sin t + (px_1 + q)\right] \frac{x_1}{x_1 - x_o} (1 - k_o x_o) - 1$$

$$= \left[(ax_1 + b) \sin t + (px_1 + q)\right] \frac{x_o(1 - k_o x_1)}{x_1 - x_o} \quad \text{---- (6 - 18a)}$$

Case II - A system of Two Ponds, both assumed perfectly mixed.



Here, from (6 -17)

$$\frac{dx_2}{dt} = \frac{E_2 x_2}{h_2} (1 - k_2 x_2) - \frac{F_1}{A_2 h_2} \frac{1 - k_2 x_2}{1 - k_1 x_1} (x_2 - x_1)$$

$= 0$ since $x_2 = x_{sat}$ = constant

$$\therefore \frac{E_2 x_2}{h_2} (1 - k_2 x_2) = \frac{F_1}{A_2 h_2} \frac{1 - k_2 x_2}{1 - k_1 x_1} (x_2 - x_1)$$

$$F_1 = \frac{A_2 E_2 x_2}{(x_2 - x_1)} (1 - k_1 x_1)$$

i.e. $$F_1 = \left[(ax_2 + b) \sin t + (px_2 + q)\right] \frac{A_2 x_2}{(x_2 - x_1)} \cdot (1 - k_1 x_1) \quad \text{---(6-19)}$$

which relates F_1 to t and x_1 as unknowns and x_2 ($= x_{saturated}$), A_2 etc. as knowns. For solution of (6-19) it is necessary to relate x_1 to t (and other knowns) and F_1 to F_o (or F_2 since both are required)

From equation (5-3a), for h, constant,

$$F_o = F_1 + E_1 A_1 \quad \text{---- (6 - 20)}$$

$$\text{and } F_1 = F_2 + E_2 A_2 \quad \text{---- (6 - 21)}$$

From equation - (6-15)

$$\frac{dx_1}{dt} = \frac{E_1 x_1}{h_1} (1 - k_1 x_1) - \frac{F_o}{A_1 h_1} \left[\quad k_1 x_1)/(1 - k_o x_o) \right] (x_1 - x_o)$$

$$= \left[(ax_1 + b) \sin t + (px_1 + q)\right] \frac{x_1}{h_1} (1 - k_1 x_1) - \frac{F_o}{A_1 h_1} \frac{1 - k_1 x_1}{1 - k_o x_o} \cdot (x_1 - x_o)$$

and substituting from equation (6 - 20)

$$\frac{dx_1}{dt} = \left[(ax_1 + b)\sin t + (px_1 + q)\right]\frac{x_1}{h_1}(1-k_1x_1) - \frac{(F_1+E_1A_1)}{A_1h_1}\left[\frac{1-k_1x_1}{1-k_ox_o}\right](x_1 - x_o)$$

---- (6 - 22)

Substitution for F_1 from (6-19) and for E_1 from (3-27) gives

$$\frac{dx_1}{dt} = \left[(ax_1 + b)\sin t + (px_1 + q)\right]\frac{x_1}{h_1}(1-k_1x_1) - \frac{(1-k_1x_1)}{(1-k_ox_o)}\cdot\frac{(x_1 - x_o)}{h_1}$$
$$- \left[(ax_2 + b)\sin t + (px_2+q)\right]\frac{A_2x_2}{(x_2-x_1)}(1-k_1x_1)\frac{(1-k_1x_1)}{(1-k_ox_o)}\cdot\frac{(x_1-x_o)}{A_1h_1} \quad --(6\text{-}22\ a)$$

A solution of this differential equation will relate x_1 to t in terms of a, b, p, q, h_1, k_1, k_o, x_o, x_2, A_1 and A_2 all of which can be defined and consequently are known.

That is $x_1 = f(t)$ only.

Further, by simultaneous solution of equations (6-19) and (6-20) expressions for F_o, the in flow of sea water and F_2, the out flow of saturated brine, for this two pond system can be established.

viz: $$F_o = A_1\left[(ax_1 + b)\sin t + (px_1 + q)\right] + A_2\left[(ax_2 + b)\sin t + (px_2 + q)\right]\left(\frac{x_2}{x_2 - x_1}\right)(1 - k_1x_1)$$

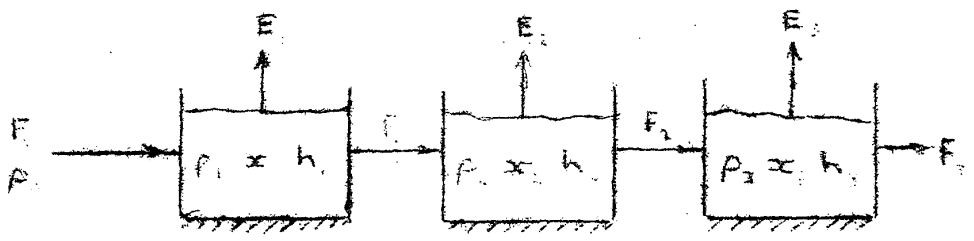
---- (6 - 23)

$$F_2 = A_2\left[(ax_2 + b)\sin t + (px_2 + q)\right]\left[\frac{x_2}{x_2 - x_1}(1 - k_1x_1) - 1\right]$$

$$= A_2\left[(ax_2 + b)\sin t + (px_2 + q)\right]\frac{x_1(1 - k_1x_2}{x_2 - x_1} \quad ---- (6 - 24)$$

Both of these equations can be solved to give F_o and F_2 as functions of t only when equation (6-22) is solved for $x = f(t)$

Case III - A system of three ponds.



Here $A_1 + A_2 + A_3$ = total area A

A similar procedure gives:

$$\frac{dx_3}{dt} = \frac{E_3 x_3}{h_3}(1 - k_3 x_3) - \frac{F_2}{A_3 h_3}(x_3 - x_2)\ \frac{1 - k_3 x_3}{1 - k_2 x_2}$$

$$= 0 \quad \text{since } x_3 = x_{sat} = \text{constant}$$

$$\therefore F_2 = \left[(ax_3 + b)\sin t + (px_3 + q)\right] \frac{A_3 x_3}{x_3 - x_2}(1 - k_2 x_2) \quad \text{--- (6 - 25)}$$

But $F_3 = F_2 - E_3 A_3$ for h_3 constant

$$= A_3 \left[(ax_3 + b)\sin t + (px_3 + q)\right] \frac{x_2(1 - k_2 x_3)}{x_3 - x_2} \quad \text{--- (6 - 26)}$$

and from (6 - 25) (6 - 20) and (6 - 21) -

$$F_1 = F_2 + E_2 A_2$$

$$= \frac{E_3 A_3 x_3}{x_3 - x_2}(1 - k_2 x_2) + E_2 A_2 \quad \text{--- (6 - 27)}$$

$$F_o = F_1 + E_1 A_1$$

$$= \frac{E_3 A_3 x_3}{x_3 - x_2}(1 - k_2 x_2) + E_2 A_2 + E_1 A_1$$

$$= A_3 \left[(ax_3 + b)\sin t + (px_3 + q)\right] \frac{x_3}{x_3 - x_2}(1 - k_2 x_2)$$
$$+ A_2 \left[(ax_2 + b)\sin t + (px_2 + q)\right]$$
$$+ A_1 \left[(ax_1 + b)\sin t + (px_2 + q)\right] \quad \text{--- (6 - 28)}$$

From equations (6-26) and (6-28) it is clear that both x_1 and x_2 must be expressed as functions of t which apply simultaneously.

From equation (6-16) -

$$\frac{dx_2}{dt} \equiv \frac{E_2 x_2(1 - k_2 x_2)}{h_2} - \frac{F_1}{A_2 h_2}\frac{(1 - k_2 x_2)}{(1 - k_1 x_1)}(x_2 - x_1)$$

and substitution for F_1 and E_2 into this equation from equations (6-27) and (3-27) gives :

$$\frac{dx_2}{dt} = \left[(ax_2 + b)\sin t + (px_2 + q)\right]\left[\frac{x_2(1 - k_2 x_2)}{h_2} - \frac{(1-k_2 x_2)(x_2 - x_1)}{h_2(1-k_1 x_1)}\right]$$
$$-\frac{A_3 x_3}{(x_3 - x_2)} \cdot \frac{(1-k_2 x_2)^2}{A_2 h_2}\frac{(x_2 - x_1)}{(1-k_1 x_1)}\left[(ax_3 + b)\sin t + (px_3 + q)\right]$$
$$\text{--- (6 - 29)}$$

For x_1 in terms of t from (6-22)

$$\frac{dx_1}{dt} = \frac{E_1(1 - k_1x_1)}{h_1}\left[x_1 - \frac{x_1-x_o}{1-k_ox_o}\right] - \frac{F_1}{A_1h_1}\frac{(1-k_1x_1)}{(1-k_ox_o)}(x_1 - x_o)$$

$$= \frac{(1 - k_1x_1)}{h_1(1-k_ox_o)}\left[E_1x_o(1-k_ox_1) - \frac{F_1}{A_1}(x_1 - x_o)\right] \quad \text{---- (6 - 30)}$$

and substitution for F_1 from (6-27) gives -

$$\frac{dx_1}{dt} = \frac{(1-k_1x_1)}{h_1(1-k_ox_o)}\left\{E_1x_o(1-k_ox_1) - \left[\frac{E_3A_3x_3}{(x_3-x_2)}(1-k_2x_2) + E_2A_2\right]\frac{x_1-x_o}{A_1}\right\}$$

and substitution for E_1, E_2, E_3 from (3-27) gives -

$$\frac{dx_1}{dt} = \frac{(1-k_1x_1)}{h_1(1-k_ox_o)}\left\{\left[(ax_1+ b)\sin t + (px_1 + q)\right]x_o(1 - k_ox_o)\right.$$

$$- \left[(ax_3+ b)\sin t + (px_3+q)\right]\frac{A_3}{A_1}x_3\frac{x_1- x_o}{x_3- x_2}(1 - k_2x_2)$$

$$\left. - \left[(ax_2+ b)\sin t + (px_2+ q)\right]\frac{A_2}{A_1}(x_1-x_o)\right\} \quad \text{---- (6-31)}$$

Equations (6-29) and (6-31) both contain the terms x_1, x_2, x_3 so that they must be solved simultaneously to give the expressions for x_1 and x_2 in terms of t and knowns.

Case IV - A General system of n ponds.

For pond (n-1), since $x_n = x_{sat}$,

$$\frac{dx_{n-1}}{dt} = \left[\frac{x_{n-1}(1-k_{n-1}x_{n-1})}{h_{n-1}} - \frac{(1-k_{n-1}x_{n-1})(x_{n-1} - x_{n-2})}{h_{n-1}(1-k_{n-2}x_{n-2})}\right]E_{n-1}$$

$$- \left[\frac{A_nx_n}{(x_n - x_{n-1})}\cdot\frac{(1-k_{n-1}x_{n-1})^2}{A_{n-1}h_{n-1}}\cdot\frac{(x_{n-1} - x_{n-2})}{1 - k_{n-2}x_{n-2}}\right]E_n$$

For all other ponds in the system of n ponds -

$$\frac{dx_r}{dt} = \frac{E_rx_r(1 - k_rx_r)}{h_r} - \frac{F_{r-1}}{A_rh_r}\cdot\frac{1 - k_rx_r}{1 - k_{r-1}x_{r-1}}(x_r - x_{r-1}) \quad \text{---- (6-33)}$$

Where the r th pond is any one pond and

$$\text{Where } F_{r-1} = \frac{E_nA_nx_n}{x_n - x_{n-1}}(1 - x_{n-1}k_{n-1}) + \sum_{s=r}^{s=(n-1)} E_sA_s \quad \text{---- (6-34)}$$

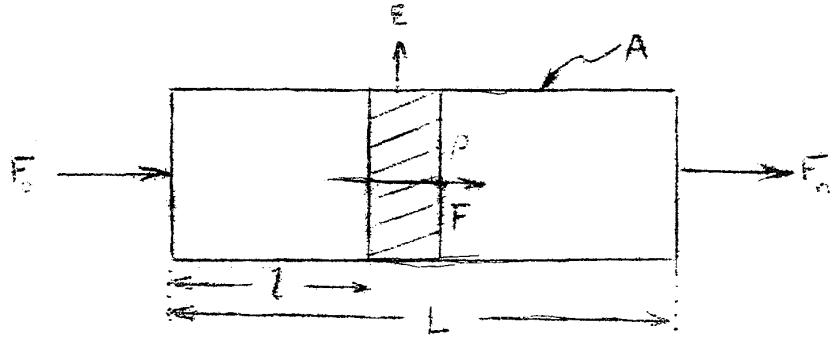
And for a solution of F_o and F_n, all the equations (6-33), after

substitution from equations (6-34), must be solved simultaneously for $x_1, x_2, x_3 \text{ -----} x_r \text{ -----} x_{n-1}$ as functions of time.

Case V - A system of an infinite number of ponds in a finite area.

This is the case of a single pond in which streamline flow takes place.

Consider such a system as defined in the accompanying figure:



Pond of area A depth h, proceeding as before.

(i) Total vol. balance

$$F - \delta F = F + E\,\delta A + \delta A \frac{\partial h}{\partial t}$$

$$= F + E\,\delta l \frac{A}{L} + \frac{A}{L}\,\delta l \frac{\partial h}{\partial t}$$

In the limit -

$$\therefore -\frac{\partial F}{\partial l} = \frac{EA}{L} + \frac{A}{L}\frac{\partial h}{\partial t} \qquad \text{and if } h \text{ is correct}$$

$$-\frac{\partial F}{\partial l} = \frac{AE}{L} \qquad \text{---------- (6-35)}$$

(ii) Weight balance.

$$(F - \delta F)(\rho - \delta\rho) = F\rho + E\,\delta A\,\rho_w + \frac{\partial w}{\partial t}$$

$$= F\rho + E\frac{A}{L}\delta l \rho_w + \frac{A}{L}\delta l \frac{\partial}{\partial t}(h\rho)$$

$$\therefore - F\delta\rho - \rho\delta F = \delta l \frac{A}{L} E\rho_w + \frac{\partial}{\partial t}(h\rho)$$

$$-\delta(F\rho) = \delta l \frac{A}{L} E\rho_w + \frac{\partial}{\partial t}(h\rho)$$

Thus $$-\frac{\partial(F\rho)}{\partial l} = \frac{A}{L} E\rho_w + \frac{\partial}{\partial t}(h\rho) \qquad \text{---- (6 - 36)}$$

(iii) NaCl balance.

$$(F - \delta F)(\rho - \delta\rho)(x - \delta x) = F\rho\, x + \frac{\partial W_{NaCl}}{\partial t}$$

$$= F\rho\, x + \frac{\partial}{\partial t}\left(h.\frac{A}{L}.\ \delta l.\ \rho x\right)$$

$$\therefore\ -\delta(F\rho x) = \frac{A}{L}\,\delta l\ \frac{\partial}{\partial t}\ (h\rho x)$$

$$\text{or}\ \frac{-\partial}{\partial l}(F\rho x) = \frac{A}{L}\ \frac{\partial}{\partial t}(h\,\rho x) \qquad \text{---- (6 - 37)}$$

$$= \frac{A}{L}\, h\,\rho\,\frac{\partial x}{\partial t} + x\ \frac{\partial(h\,\rho)}{\partial t}$$

$$\text{and from Equation (6-36)} = \frac{Ah\rho}{L}.\ \frac{\delta x}{\partial t} - x\ \frac{\partial(F\rho)}{\partial l} + \frac{A}{L}.\ E\rho_w$$

$$-x\frac{\partial(F\rho)}{\partial l} - F\rho\,\frac{\partial x}{\partial l} = \frac{Ah\rho}{L}.\ \frac{\partial x}{\partial t} - x\ \frac{\partial(F\rho)}{\partial l} + \frac{A}{L}.\ E\rho_w$$

$$\therefore\ \frac{Ah\rho}{L}.\ \frac{\partial x}{\partial t} = x.E.\frac{A}{L}\ \rho w - F\rho\ \frac{\partial x}{\partial l}$$

$$\frac{\partial x}{\partial t} = \frac{Ex}{h}.\ \left(\frac{\rho w}{\rho}\right) - \frac{FL}{Ah}.\ \frac{\partial x}{\partial l} \qquad \text{---- (6 - 38)}$$

$$\text{i.e.}\quad \frac{\partial x}{\partial t} = \frac{Ex}{h}\ (1 - kx) - \frac{FL}{Ah}\ \frac{\partial x}{\partial l} \qquad \text{---- (6 - 39)}$$

It is clear from equation (6-39) that x has become a function of time and of the fraction of the distance the fluid has passed through the system. (Viz: $^{l}/_{L}$)

Consequently, x cannot be solved for by equation (6-39) alone, which means that here, as for the general case, no general solution is possible.

To simplify the case, it was postulated that for an infinite number of ponds, $\frac{\partial x}{\partial t} = 0$ for all x and t. This postulate was made after considering the cases of finite numbers of ponds, for which there are limits to the values x can take in each pond, so that cycling is allowable. (See later). These limits decrease as the number of ponds increases, so that by extrapolation it was suggested that for an infinite number of ponds, the x value would remain very nearly constant for all t i.e. $\frac{dx}{dt}$ becomes very small and hence postulate $\frac{dx}{dt} = 0$. This postulate has not been proved to the satisfaction of the author and there appears some difficulty in establishing its truth.

Application of this postulate leads to a complex statement:

Differentiating equation (6-39) w.r.t. t gives -

$$\frac{\partial^2 x}{\partial t\,\partial l} = \frac{1}{h} \cdot \frac{\partial E}{\partial x} \cdot \frac{\partial x}{\partial l} \left[x - kx^2 \right] + \frac{E}{h} (1-2kx) \frac{\partial x}{\partial l}$$

$$- \frac{L}{Ah} \cdot F \cdot \frac{\partial^2 x}{\partial l^2} - \frac{L}{Ah} \cdot \frac{\partial F}{\partial l} \cdot \frac{\partial x}{\partial l} \quad ---- \quad (6-40)$$

Let $\frac{\partial x}{\partial t} = 0$ and hence $\frac{\partial^2 x}{\partial t \partial l} = 0$

so that in (6-40), given also

$$- \frac{\partial F}{\partial l} = + \frac{EA}{L} \quad ---- \quad (6-35)$$

$$0 = \frac{\partial E}{\partial x} \cdot \frac{\partial x}{\partial l} \left[x - kx_2^2 \right] + E \left[1-2kx \right] \frac{\partial x}{\partial l}$$

$$- \frac{L}{A} \cdot F \cdot \frac{\partial^2 x}{\partial l^2} + \frac{L}{A} \cdot \frac{EA}{L} \cdot \frac{\partial x}{\partial l}$$

But, rearranging equation (6-39), for -

$\frac{\partial x}{\partial t} = 0$, gives $F = \frac{Ah}{L} \cdot \frac{Ex}{h} (1 - kx)$, $\Big/ \frac{\partial x}{\partial l}$

Hence $0 = x(1-kx) \cdot \frac{\partial E}{\partial x} \cdot \frac{\partial x}{\partial l} + E \frac{\partial x}{\partial l} (1-2kx + 1)$

$$- Ex(1 - kx) \frac{\partial^2 x}{\partial l^2} \Big/ \frac{\partial x}{\partial l}$$

But $\frac{\partial x}{\partial l} \neq 0$ and in the practical range of x for this problem

viz: $0.0199 = x = 0.137$, $(1 - kx) = 0$

Hence:- $Ex \frac{\partial^2 x}{\partial l^2} - (2E + x \frac{\partial E}{\partial x}) (\frac{\partial x}{\partial l})^2 = 0 \quad ---- \quad (6-41)$

This equation relates x to l for any t (since E is a function of t) but for x to be independent of t, i.e. $\frac{dx}{dt} = 0$, there must be no term in t for the expression of x in terms of l.

Hence, from equations (6-41) and (3-39)

$$(ax + b) \sin t \; x \frac{\partial^2 x}{\partial l^2} - 2 \cdot (ax + b) \sin t \cdot \frac{(\partial x)^2}{(\partial l)}$$

$$- x a \sin t \frac{(\partial x)^2}{(\partial l)} = 0$$

i.e. $(ax + b) x \frac{\partial^2 x}{\partial l^2} = \frac{(\partial x)^2}{(\partial l)} \left[3ax + 2b \right]$, when $\sin t \neq 0$

or since x is a function of l only

$$\frac{d^2x}{dl^2} = \left(\frac{dx}{dl}\right)^2 \frac{3ax + 2b}{x(ax + b)} \qquad \text{---- (6 - 42)}$$

Also from equation (6-41) the condition -

$$x\,(px + q)\,\frac{\partial^2 x}{\partial l^2} - 2(px + q)\left(\frac{\partial x}{\partial l}\right)^2 - x\,p\left(\frac{\partial x}{\partial l}\right)^2 = 0$$

$$\text{i.e.} \quad \frac{\partial^2 x}{\partial l^2} = \left(\frac{\partial x}{\partial l}\right)^2 \frac{3px + 2q}{x(px + q)} \qquad \text{---- (6 - 42a)}$$

must apply which will give a second relation of x to l independent of t for all t.

For equation (6 - 42)

let $\frac{dx}{dl} = Z$, then $\frac{d^2x}{dl^2} = \frac{dZ}{dl} = \frac{dx}{dl} \cdot \frac{dZ}{dx} = Z \cdot \frac{dZ}{dx}$

Hence in (6 - 42) above:

$$Z\frac{dZ}{dx} = Z^2 \frac{(3ax + 2b)}{(x(ax + b))} \qquad \text{---- (6 - 43)}$$

$$\text{i.e.} \quad \frac{dZ}{Z} = \frac{3ax + 2b}{x(ax + b)}\,dx = \frac{2}{x} + \frac{a}{(ax + b)}\,dx$$

Integrating,

$$\ln Z = \ln x^2 + \ln(ax + b) + \ln g$$

where g = arbitrary constant

$$\text{i.e.} \quad Z = gx^2(ax + b) = \frac{dx}{dl} \qquad \text{---- (6 - 44)}$$

To evaluate g, some boundary condition is required. Nothing is known about $\frac{dx}{dl}$ at any point in the system but the two boundary conditions of -

$$x = 0.0199 \quad \text{when} \quad l = 0$$

$$\text{and} \quad x = 0.137 \quad \text{when} \quad l = L$$

are known, which will allow evaluation of g later.

Integrating equation (6-44) -

given $a = -4.0$: $b = +3.02$; (table 3.11)

$$\int dl = \frac{1}{g} \int \frac{dx}{x^2(ax + b)} + f$$

where f = an arbitrary constant -

$$\therefore l = \frac{1}{g}\int \frac{dx}{x^2(ax+b)} + f$$

$$= \frac{1}{g}\int \left[\frac{-a}{b^2 x} + \frac{1}{bx^2} + \frac{a^2}{b^2(ax+b)}\right] dx + f$$

$$= \frac{1}{g}\left[-\frac{a}{b^2}\ln x - \frac{1}{bx} + \frac{a^2}{b^2}\ln(ax+b)\right] + f$$

$$= \frac{1}{g}\left[\ln\left(\frac{ax+b}{x}\right)^{a/b^2} - \frac{1}{bx}\right] + f \qquad \text{---- (6 - 45)}$$

Equation (6-45) only holds when $\sin t \neq 0$. For all values of t, from equation (6-42a), since x is independent of t,

$$\frac{d^2x}{dl^2} = \left(\frac{dx}{dl}\right)^2 \frac{3px+2q}{x(px+q)} \qquad \text{---- (6 - 46)}$$

Put $Z = \frac{dZ}{dx}$ whence $\frac{d^2x}{dl^2} = Z\frac{dZ}{dx}$

Hence $\frac{dZ}{dx} = Z\frac{3px+2q}{x(px+q)}$

or $\ln Z = \ln x^2 + \ln(px+q) + \ln r$

i.e. $Z = rx^2(px+q) = \frac{dx}{dl}$

i.e. $l = \frac{1}{r}\left[-\frac{p}{q^2}\ln x - \frac{1}{qx} + \frac{p}{q^2}\ln(px+q)\right] + S$

i.e. For all t:

$$l = \frac{1}{r}\left[\ln\frac{(px+q)^{p/q^2}}{x} - \frac{1}{qx}\right] + S \qquad \text{---- (6 - 47)}$$

Where r and s are arbitrary constants.

This expression is rather complex and difficult to evaluate $\frac{dx}{dl}$ for insertion into equation (6-41). Also, it will be seen on evaluating it, that no terms can be neglested. So that an attempt was made to express x as a quadratic in l.

viz: $x = rl^2 + sl + Z$ ----(6 - 48)

where r, s, Z are constants, independent of t, since x is independent of t. $(\frac{dx}{dt} = 0)$.

But when $l = 0$, $x = 0.0199$ for all t

Further $\frac{\partial x}{\partial l} = 2rl + S$ ---- (6 - 49)

$$\left(\frac{\partial x}{\partial l}\right)^2 = 4r^2l^2 + S^2 \quad \text{---- (6 - 50)}$$

$$\frac{\partial^2 x}{\partial l^2} = 2r \quad \text{---- (6 - 51)}$$

so that substitution into equation (6-41) from equations (6-49), (6-50) (6-51) and (3-39) gives:

$$\left\{a(rl^2 + sl + Z) + b\right\}\sin t + p(rl^2 + sl + Z) + q\Big\}(rl^2+sl+Z)\,2r$$
$$= \Big\{2\left\{a(rl^2 + sl + Z) + b\right\}\sin t + p(rl^2 + sl + Z) + q$$
$$+ (rl^2 + sl + Z)(a \sin t + p)\Big\}(4r^2l^2 + 4rsl + s^2) \text{----(6 - 52)}$$

Equation (6-52) can be used for equating co-efficients and hence solving for r, s given Z.

Equating coefficients in l^4 and in l^3 leads to the solution, in both cases,

of $\sin t = p/a = \text{constant}$ ---- (6 - 53)

This clearly is not a solution for all t.

for the coefficients of l^2 a result of

$$(6rb + 13\,S^2a + 8rZa)\ \sin t$$
$$+ (6rq + 13S^2p + 8rZp) = 0 \text{ is obtained}$$

and since sint $\neq$ constant for all t,

$$r = s = Z = 0 \text{ for all } t \quad \text{---- (6 - 54)}$$

This is not a solution either.

For the coefficient of l:

$$(3as^3 + 6arsZ + 4b\,rs) \sin t$$
$$+ (3ps^3 + 6\,prsZ + 4rsq) = 0$$

Hence again:- $r = s = Z = 0$

A similar solution is obtained for the term, independent of l.

Thus no solution for r and s is obtainable by this method, which suggests that application of a quadratic for x in l is not permissable and either equation (6-45) or (6-47) must be used in full.

Equation (6-47) applies for all t and will be the one used for an expression of x in terms of l. With this expression, a solution for the flow can be determined.

From equation (6-35)

$$\frac{dF}{dl} = - \frac{AE}{L}$$

$$\therefore F_n - F_o = - A\int_o^L \left[(ax + b) \sin t + (px + q)\right] d\,(l/L)$$

From equation (3 - 39)

but l is independent of t so that

$$F_n - F_o = - A\int_o^L (px + q)\, d\,(l/L) - A \sin t \int_o^L (ax+b)\, d(l/L)$$

---- (6 - 55)

Substitution for $d(l/L)$ from equation (6-47) would give an integratable function for equation (6-55). However, due to the complex nature, a graphical procedure will be employed.

Also from equation (6-36)

$$- \frac{d(F\rho)}{dl} = \frac{A}{L}\left[E\rho_w + \frac{d}{dt}(h\rho)\right]$$

assuming h is constant for all t, and since -

$$\rho = \frac{\rho_w}{1 - kx}, \quad \frac{d\rho}{dt} = 0 \text{ if } \frac{dx}{dt} = 0,$$

so that

$$-\int_{l=o}^{l=L} d(F\rho) = \rho_w A\int_o^L E\, d\,(l/L)$$

i.e. $$F_o\rho_o - F_n\rho_n = \rho_w A\int_o^L (px + q)\, d\,(l/L) + \rho_w A \sin t \int_o^L (ax + b) d\,(l/L)$$

---- (6 - 56)

Combination of equations (6-55) and (6-56) gives -

$$\rho_o F_o - \rho_n F_n = \rho_w (F_o - F_n)$$

$$\text{i.e. } F_o = \frac{\rho_n - \rho_w}{\rho_o - \rho_w} F_n$$

and hence in equation (6-55)

$$F_n \frac{\rho_n - \rho_o}{\rho_o - \rho_w} = A\int_o^L (px + q)\, d(l/L) + A \sin t \int_o^L (ax + b)\, d\,(l/L))$$

---- (6-57)

Given values of a, b, p, q (from § 3) and of ρ_o, ρ_n, ρ_w equation (6-57) can be evaluated for F_n, the output of saturated brine from a system of an infinite number of ponds, as a function of time.

Also as $F_o = \frac{\rho_n - \rho_w}{\rho_o - \rho_w} F_n$, the input of sea water can be estimated.

Solutions to cases 1 - V when average weather conditions apply.

For a one pond system.

From equation (6-18)

$$F_o = \frac{E_1 A_1 x_1}{(x_1 - x_o)} \quad (1 - k_o x_o)$$

and given $x_1 = x_{saturated} = 0.137$ --- (Table 4.1)

$x_o = x_{sea\ water} = 0.0199$

and that $E = (ax + b) \sin t + (px + q)$

where $a = -4.0$

$b = + 3.02$ -----(§ 3)

$p = - 7.12$

$q = + 1.73$

$k_o = 1.295$ --- (Table 4.0)

so that if $A_1 = A$ = total area of the system

$$\frac{F_o}{A} = \frac{(2.47 \sin t + 0.75)\ (1 - 1.295 \times 0.0199)\ 0.137}{(0.137 - 0.0199)}$$

$$= 2.82 \sin t + 0.855 \text{ in}^3/\text{in}^2/\text{month}$$

$$= 0.235 \sin t + 0.0713 \text{ ft}^3/\text{ft}^2/\text{month} \quad ---(6\text{-}58)$$

For $\frac{F_1}{A}$, from equation (6-18a)

$$\frac{F_1}{A} = 0.346 \sin t + 0.106 \text{ in}^3/\text{in}^2/\text{month}$$

$$= 0.0289 \sin t + 0.0089 \text{ ft}^3/\text{ft}^2/\text{month} \quad --- \quad (6\text{-}59)$$

Note A = area of the pond in square feet.

= total area of the system

The quantities of brine flowing can be given by:-

The total quantity flowing per unit area of system between times t_1 and $t_2 = \int_{t_1}^{t_2} \frac{F}{A}\, dt$ --- (6-60)

Thus, the total input of sea water per unit area of pond for a one pond system = Q_1

$$= \left[-0.235 \cos t + 0.0713t \right]_{t_1}^{t_2} \quad --- \quad (6\text{-}61)$$

Define $Q_1 = 0$ when $t = t_1 = 0$, so that

if $t_2 = t$, $Q_1 = +0.235(1 - \cos t) + 0.0713t$ ft^3/ft^2 area

t is in months and 1 month $\equiv \frac{\pi}{6}$ ------ (6-62)

The total output of saturated brine from the same system, is from equations (6-60) and (6-59),

$$Q_1^* = 0.0289\ (1 - \cos t) + 0.0089t \quad \text{------} \quad (6\text{-}63)$$

The evaluation of equations (6-58), (6-59) and (6-63) for $\frac{F_o}{A}$, $\frac{F_1}{A}$, and Q_1^* gives the results shown in table (6.1). The value of Q_1^* are from time t = 0, to time t = t in the table, where the unit of time is $\frac{\pi}{6} \equiv$ 1 month.

It should be noted that units of Q_1^* are $\frac{\text{ft}}{\text{month}}$ x radians

$$\therefore\ Q_1^* = \left(0.0289\ (1 - \cos t) + 0.0089t\right) \frac{6}{\pi} \quad \text{ft}^3/\text{ft}^2$$

DATE	t RADIANS	F_o/A ft/in.	F_1/A_1 ft/mon.	Q_1^* ft^3/ft^2
MID SEP.	0	$\bar{0}.0713$	$\bar{0}.0089$	0.0000
MID OCT.	$\frac{\pi}{6}$	0.1888	0.0234	0.0162
MID NOV.	$\frac{\pi}{3}$	0.2748	0.0339	0.0455
MID DEC.	$\frac{\pi}{2}$	0.3063	0.0378	0.0819
MID JAN.	$\frac{2\pi}{3}$	0.2748	0.0339	0.1184
MID FEB.	$\frac{5\pi}{6}$	0.1888	0.0234	0.1474
MID MAR.	π	0.0713	0.0039	0.1639
MID APR.	$\frac{7\pi}{6}$	$\bar{0}.0432$	$\bar{0}.0056$	0.1652
MID MAY.	$\frac{4\pi}{3}$	$\bar{0}.1322$	$\bar{0}.01613$	0.1539
MID JUNE	$\frac{3\pi}{2}$	$\bar{0}.1637$	$\bar{0}.0200$	0.1352
MID JULY	$\frac{5\pi}{3}$	$\bar{0}.1522$	$\bar{0}.01613$	0.1167
MID AUG.	$\frac{11\pi}{6}$	$\bar{0}.0462$	$\bar{0}.0056$	0.1052

Solution for the two pond case

For the two pond case, a solution is required of equation (6-22a) to give a relation between x_1 and t.

i.e. $$\frac{dx_1}{dt} = E_1 \left[\frac{x_1}{h_1} (1-k_1x_1) - \frac{(1-k_1x_1)}{(1-k_ox_o)} \frac{(x_1-x_o)}{h_1} \right]$$

$$- E_2 \left[\frac{A_2x_2 (1-k_1x_1)^2 (x_1 - x_o)}{A_1h_1 (1-k_ox_o) (x_2 - x_1)} \right]$$

where E_1 and E_2 are given as functions of x in equations (3-39)

$$\frac{dx_1}{dt} = \frac{(1-k_1x_1)}{h_1}\left[E_1\left(x_1 - \frac{x_1-x_o}{1-k_ox_o}\right) - E_2\frac{A_2}{A1}\frac{x_2(1-k_1x_1)(x_1-x_o)}{(x_2-x_1)(1-k_ox_o)}\right]$$

$$=\frac{(1-k_1x_1)}{h_1(1-k_ox_o)}\left[E_1\ x_o(1-k_ox_1) - E_2\frac{A_2}{A_1}\frac{x_2(x_1-x_o)}{(x_2-x_1)}\ (1-k_1x_1)\right]$$

From table 4.0), values of k for various values of x show that $k_o \doteqdot k_1 = k$ ca 1.27 on an average. It will be assumed, k is constant throughout the system equal to 1.27.

$$\therefore\ \frac{dx_1}{dt} = \frac{(1-kx_1)^2}{h_1(1-kx_o)}\left[x_o(x_2-x_1)\,E_1 - \frac{A_2}{A_1}x_2(x_1-x_o)\ E_2\right] \qquad ---(6\text{-}64)$$

where $E_1 = (ax_1+b)\sin t + (px_1+q)$

$E_2 = (ax_2+b)\sin t + (px_2+q)$ ----(3-39)

$x_2 = x_{saturated} = 0.137$

$x_o = x_{sea\ water} = 0.0199$

The values of a , b , p , q are given in §§ 3 and

$k = 1.27$

Let $\frac{A_2}{A_1} = \gamma$ (say)

h_1 = depth of brine in pond 1 in inches.

Inserting these values gives when $x_1 = x$

$$\frac{dx_1}{dt} = \frac{1}{h_1}\frac{(1-1.17x)^2}{(0.137-x)}\left\{\begin{array}{l}\left[(0.08166)x^2 - (0.07284 + 0.34744\gamma)\,x\right.\\ \left.(0.00844 + 0.00691\gamma)\right]\sin t\\ +\ 0.14536x^2 - (0.05523 + 0.10605\gamma)\,x\\ +\ (0.00484 + 0.00211\gamma)\end{array}\right\} \text{--- (6-65)}$$

In the special case where $\gamma = 1$, i.e. $A_2 = A_1$, this reduces to

$$\frac{dx}{dt} = \frac{1}{h_1}\frac{1-1.27x)^2}{(0.137-x)}\left[\begin{array}{l}(0.08166x^2 - 0.42028x + 0.01536)\sin t\\ +\ (0.14536x^2 - 0.16129x + 0.00695)\end{array}\right] \text{--- (6.66)}$$

This equation is clearly difficult to solve so some estimate of the most likely value of γ should be made.

Differentiating equation (6.65) w.r.t. γ , at any given x and t:

$$\frac{(0.137-x)}{(1-1.27x)^2}\cdot h_1\left[\frac{\partial}{\partial\gamma}\cdot\frac{dx}{dt}\right]_{x_1t} = \left[\begin{array}{l}(-0.33866x + 0.006737)\sin t\\ +\ (-0.10337x + 0.002053)\end{array}\right]$$

i.e. at any given value of x and t

$$\left(\frac{\partial}{\partial \gamma} - \frac{dx}{dt}\right)_{xt} = k \text{ (a constant)}$$

and integrating

$$\Delta\left(\frac{dx}{dt}\right) = (\Delta \gamma)\, k \qquad \text{--- (6-67)}$$

Thus, if in a system, was changed at any given x and t, then the new slope, S_2, of the x t curve would be related to the old slope S_1 and γ by $S_2 = S_1 + \Delta\left(\frac{dx}{dt}\right) = S_1 + k(\Delta\gamma)$ --- (6-68)

For the order of changes in $\frac{(dx)}{(dt)}$ due to changes in γ, it will be shown later that $x = 0.05$ is a reasonable average value of x, Further, maximum values in x occur at about:-

$t = \frac{11\pi}{6}$ so $\sin t \sim 0.5$ and $\frac{dx}{dt} \sim -0.004$ for $\gamma = 1$

$$\therefore \quad k \sim -\frac{0.01 \sin t + 0.003}{0.09\, xh/0.9} \quad \text{and at } h = 12"$$

$$\sim -0.08/12 \sim -0.007$$

Say $k \sim -0.01$ --- (6-69)

$\therefore$ $S_2 = S_1 - 0.01\ \Delta\gamma$ so that for an initial value of $\gamma = 1$,

$$S_2 \sim -0.004 - 0.01\ (\Delta\gamma) \qquad \text{--- (6-70)}$$

(i) Let γ increase, i.e. $A_2 > A_1$, so that $\Delta\gamma = 1$ i.e. $A_2 = 2A_1$

$\therefore$ $S_2 \sim -0.004 - 0.01$

~ -0.014

Increasing A_2, in size, above A_1 to the maximum limit when A_2 = total area, makes $A_1 = 0$ & $\Delta\gamma = \infty$

$\therefore$ new slope $S_2 \sim \infty$

(ii) Decrease γ so that the ratio of $A_2 : A_1$ is reversed, i.e. $A_2 = \frac{1}{2}A_1$ whence $\Delta\gamma = -0.5$

$\therefore$ $S_2 \sim -0.004 + 0.005 \sim +0.001$

The limit on decreasing A_2 in size compared with A_1 makes $A_2 = 0$ $\Delta\gamma = -1$ & $S_2 \sim -0.004 + 0.01 \sim +0.006$

Comparing the cases (i) and (ii), it is clear that with changing ratio of the areas of the first and second ponds, in a system

of two ponds, the slope of the $x\ t$ curve at a particular value of x and t is greatly affected.

(i) an increase in area from pond 1 to pond 2 is associated with large fluctuations in the slope of the $x\ t$ curve. The limit of this process ($A_1 = 0$) gives an infinite slope to the curve. Thus, the $x\ t$ curve would oscillate rapidly and control of the system would be hard to maintain. An unusual spell of weather would seriously affect the flow of brines so that only saturated brines flowed from the system.

(ii) A decrease in area from pond 1 to pond 2 gives changes in the slope of the $x\ t$ curve only from negative to positive slope of the same order, even in the limiting case of $\Delta\gamma = -1$, $A_2 = 0$.

Such vigorous oscillations as in (i) will be absent and control over the system far easier to maintain.

Thus, in a system of two ponds A_2 should not be made greater than A_1 since as the ratio A_2/A_1 increases, control over the system becomes harder to maintain.

The effect over a period of time, of altering the ratios of $A_2 : A_1$ on output of saturated brine can only be completely established by solving equation (6-65) for x, and hence F_2, as a function of time for various values of γ. Instantaneously, however, considering two systems containing two ponds each:

(a) $A_2 = kA_1$ where $k > 1$

(b) $A_1 = kA_2$

and in both cases, the total area

$$A = (1 + k)\, A_1$$

Instantaneously let the flow rate of sea water into each system be F_a and F_b, where F_a and F_b are such that saturated brine leaves each system. Assume the concentration in pond 1 and in pond 2 are the same in each case at the instant considered. Then the total evaporation from each system at that instant is, from Fig. (6.1)

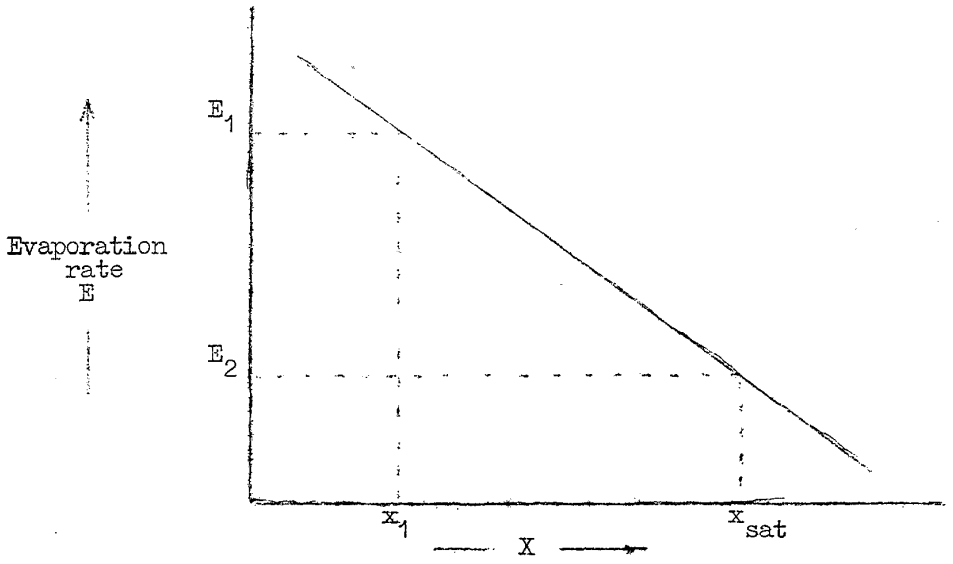


Figure (6.1) - Evaporation rate versus brine concentration curve.

(a) $E_1A_1 + E_2A_2 = E_1A_1 + E_2kA_1$

(b) $E_1kA_1 + E_2A_1$

and the flow out of brine is:-

(a) $F_a - E_1A_1 - E_2kA_1$

(b) $F_b - E_1A_1k - E_2A_1$

Then for equal flows of sea water in i.e. $F_a = F_b$, the output of (a) $\gtrless$ output of (b)

if $F_a - E_1A_1 - E_2kA_1 \lessgtr F_b - E_1kA_1 - E_2A_1$

i.e. if $F_a - F_b \lessgtr A_1\left[E_1(1-k) - E_2(1-k)\right]$

i.e. if $0 \lessgtr A_1\left[E_1 - E_2\right]\left[1-k\right]$

i.e. if $E_2 \lessgtr E_1$

but $x_2 = x_{saturated}$ and so is always greater than x_1 $\therefore$ $E_2 < E_1$ at all times.

$\therefore$ output of system (a) < output from system (b)

Thus, at any instant of time, in two separate systems of two ponds each, the output of saturated brine is greater for the system where $A_2 < A_1$ than for the system where $A_1 < A_2$ when, at that instant,

$$(x_1)_a = (x_1)_b \quad \text{and}$$

$$F_a = F_b$$

To maintain the flows the input of sea water into system (b) i.e. F_b must increase. The outputs then become related only in a complex manner and can only be solved by solution of F_2 as a function of t for various values of γ.

From instantaneous considerations it appears, for the two pond case, $A_2 < A_1$ would give better control and greater output of saturated brine from the total area. A detailed investigation would be required and the case of $\gamma = 1$ will be considered first.

Solution of the equation,

$$\frac{dx}{dt} = \frac{1}{h} \frac{(1-1.27x)^2}{(0.137-x)} \left[\begin{array}{l} (0.08166x^2 - 0.42028x + 0.01536) \sin t \\ + (0.14536x^2 - 0.16129x + 0.00695) \end{array} \right]$$

for x as a function of t

This equation can be written as:-

$$\frac{dx}{dt} = \frac{1}{h} \frac{(1-1.27x)^2}{(0.137-x)} \left[\Phi_1(x) \sin t + \Psi_1(x) \right] \qquad \text{--- (6-71)}$$

where $\Phi_1(x) = 0.08166x^2 - 0.42028x + 0.01536$

$\Psi_1(x) = 0.14536x^2 - 0.16129x + 0.00695$

A number of methods of solution were attempted:

(a) Standard Methods.

It is found that this equation is not of standard form and, consequently there exists no standard solution.

The methods of finding an integrating factor -

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and of finding a particular integral and a complementary function were applied.

In the latter case:

Let $\Psi_1(x) = 0$ for all t

Then $\frac{dx}{dt} = \frac{1}{h} \frac{(1-1.27x)^2}{(0.137-x)} \cdot \Phi(x) \sin t$ from which the variables can be separated giving:-

$$\frac{(0.137 - x)dx}{(1-1.27x)^2 \cdot \Phi_1(x)} = \frac{1}{h} \sin t \, dt$$

and integration of this expression gives a type of particular integral:

$$- \text{cost} = \frac{12.173}{(1 - 1.27x)} + 153.75 \ln (1-1.27x) - 45.267 \ln (5.09 x)$$
$$- 108.48 \ln (x - 0.053) + k$$

or $x = f(x)\, e^{-\text{cost}}$ where $f(x)$ is some complex function of x ---(6-72)

This immediately suggests that $e^{-\text{cost}}$ is part of an integrating factor. Further, to find the complimentary function for the equation (6-71) appears too complex, and so the method of an integrating factor was attempted.

Equation (6-71) may be written:-

$$X(x) \frac{dx}{dt} - (\Phi_1(x) \sin t + \Psi_1(x) = 0 \qquad \text{--- (6-73)}$$

where $X(x) = \frac{h\,(0.137-x)}{(1-1.27x)^2}$

which is of the form:-

$F = M + N \frac{dt}{dx} = 0$ where M, N are functions of x & t

If λ be an integrating factor such that

$\lambda F = \lambda M + \lambda N \frac{dx}{dt} = 0$ is a perfect differential equation

i.e. $\frac{d}{dt} (\lambda F) = 0$

then $\frac{\partial (M\lambda)}{\partial x} = \frac{\partial (N\lambda)}{\partial t}$ (Piagio ----

i.e. $M \frac{\partial \lambda}{\partial x} + \lambda \frac{\partial M}{\partial x} = N \frac{\partial \lambda}{\partial t} + \lambda \frac{\partial N}{\partial t}$ --- (6-74)

Applying this to equation (6-73)

where $N = X(x) = \frac{h(0.137 - x)}{(1-1.27x)^2}$ and

$$M = \Phi_1(x) \sin t + \Psi_1(x)$$

we note $\frac{\partial N}{\partial t} = 0$ and

$$\frac{h(0.137-x)}{(1-1.27x)^2} \cdot \frac{\partial \lambda}{\partial t} = - \left[\Phi_1(x) \sin t + \Psi_1(x) \right] \frac{\partial \lambda}{\partial x}$$
$$- \lambda \frac{\partial}{\partial x} \left[\Phi_1(x) \sin t + \Psi_1(x) \right] \qquad \text{--- (6-75)}$$

From (6-72), $e^{-\text{cost}}$ is part of λ, but equation (6-75) must be solved for the complete λ. The solution of this appears no easier than that of the original equation for $\frac{dx}{dt}$.

Consequently, a direct solution by a standard method seems impractical.

(b) Series Solution

If c is any integer such that $c \geq 0$,

Let $x = t^c \left[a_o + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \text{------} \right]$ ---(6-76)

$\therefore \frac{dx}{dt} = ct^{c-1}(a_o + a_1 t + a_2 t^2 + \text{---}) + t^c(a_1 + 2a_2 t + 3a_3 t^2 + \text{---})$ --(6-77)

Equating the coefficient of t^{c-1}, $\quad 0 = a_o c t^{c-1}$

and since $a_o \neq 0$ to give a real series, $\quad c = 0$

$\therefore x = a_o + a_1 t + a_2 t^2 + a_3 t^3 + \text{-----}$ --- (6-78)

$\frac{dx}{dt} = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^4 + \text{----}$ --- (6-79)

$x^2 = a_o^2 + 2a_o a_1 t + (a_1^2 + 2a_o a_2)t^2 + 2(a_o a_3 + a_1 a_2)t^3$

$+ (2a_o a_4 + 2a_1 a_3 + a_2^2)t^4 + \text{-----}$

$x^3 = a_o^3 + 3a_o^2 a_1 t + 3a_o(a_o a_2 + a_1^2)t^2 + (a_1^3 + 6a_o a_1 a_2 + 3a_o^2 a_3)t^3$

$+ 3(a_o^2 a_4 + a_o a_2^2 + a_1^2 a_2 + 2a_o a_1 a_3)t^4 + \text{-----}$

$x^4 = a_o^4 + 4a_o^3 a_1 t + 2a_o^2 (2a_o a_2 + 3a_1^2)t^2$

$+ 4a_o(a_1^3 + 3a_o a_1 a_2 + a_o^2 a_3)t^3$

$+ (4a_o^3 a_1 + 6a_o^2 a_2^2 + 6a_o a_1 \quad a_1 a_2 + a_o a_3 \quad + a_1^4)t^4 + \text{---------}$

Further, sin t can be replaced by a series in t viz:

$$\sin t = t - \frac{t^3}{L3} + \frac{t^5}{L5} - \frac{t^7}{L7} + \text{--------} \qquad \text{--- (6-80)}$$

Substitution of these series into equation (6-71) rewritten as

$$(0.137 - x)\frac{dx}{dt} = \frac{1}{h}\left[\begin{array}{l} (0.13171x^4 - 0.88529x^3 + 1.1739x^2 \\ \qquad - 0.45929x + 0.01536)\sin t \\ +(0.23445x^4 - 0.62936x^3 + 0.56625x^2 \\ \qquad -0.17894x + 0.00695) \end{array}\right]$$

and equating coefficients, up to terms to the power of 4 gives:

Coefficient of the constant term:

$$a_1 h(0.137 - a_o) = 0.23445\, a_o^4 - 0.62936 a_o^3 + 0.56625 a_o^2 - 0.17894 a_o + 0.00695$$

$$\text{i.e.} \quad a_1 = \frac{\left[0.23445 a_o^4 - 0.62936 a_o^3 + 0.56625 a_o^2 - 0.17894 a_o + 0.00695\right]}{h(0.137 - a_o)} \qquad \text{--- (6-81)}$$

from which, given h and a_o, a_1 can be calculated.

Coefficients of the terms in t:

$$\left[0.137.2a_2 - (a_o.\ 2a_2\ A\ a1^2)\right] h$$

$$= 4a_o^3 a_1 (0.23445) - (0.62936)\, 3a_o^2 a_1 + (o.56625)\, 2a_o a_1 - (0.17894)a_1 + 0.01536$$

$$\text{i.e.} \quad a_2(0.274 - 2a_o)\ h = 0.93780 a_o^3 a_1 - 1.888\ a_o^2 a_1 + 1.13250 a_o a_1 - 0.17894 a_1 + h a_1^2 + 0.01536 \qquad \text{--- (6-82)}$$

And knowing a_o, h and a_1 from (6-81), a_2 can bec alculated.

Coefficients of the terms in t^2:

$$(0.411 - 3a_o)\ a_3 h$$

$$= a_o^3(0.52684 a_1 + 0.93780 a_2) + a_o^2 (0.70335 a_1^2 - 2.6559 a_1 - 1.88808 a_2) + a_o\ (2.3478 a_1 - 1.88808 a_1^2 + 1.1325 a_2)$$

$$(3a_2 h - 0.45929)\ a_1 - 0.17894 a_1^2 + 0.56625 a_1^2$$

Hence a_3 and similarly for a_4 (etc) to

Coefficients of terms in t^4:

$$h a_5\ (0.685 - 5a_o)$$

$$= +(0.13171) 4a_o\ (a_o^2 a_3 + 3a_o a_1 a_2 + a_1^3 + 4\, a_o^2 a_1) - (0.88529)\ (a_1^3 + 6a_o a_1 a_2 + 3a_o^2 a_3 - 3a_o^2 a_1)$$

$$+(1.17394)\ 2\ (a_o a_3 + a_1 a_2 + a_o a_1) - (0.45929)\ (a_3 + a_1)$$

$$+ (0.23445)\ \ 4a_o^3 a_4 + 6a_o^2 a_2 + 12a_o a_1 (a_1 a_2 + a_o a_3) + a_1^4$$

$$- (0.62936)\ 3\ (a_o^2 a_4 + a_o a_2^2 + a_1^2 a_2 + 2a_o a_1 a_3)$$

$$+ (0.56625)\ 2\ (a_o a_4 + a_1 a_3 + a_2^2) \quad - (0.17894)\ a_4 \text{--hence } a_5$$

It is clear that all these coefficients can be evaluated once a value of a_o is defined, i.e. a_o is the arbitrary constant of integration of equation (6-71) and this degree of freedom must be defined by some given value of x at a given time. This can be shown, for:-

$$\frac{dx}{dt} = X(x) \left[\Phi_1 (x) \sin t + \Psi_1 (x) \right]$$

$$= \Phi(x) \sin t + \Psi(x)$$

Let $t = T + T_o$ where T_o is some fixed constant value of t

Then $\frac{dx}{dt} = \Phi (x) \sin(T + T_o) + \Psi(x)$ ------- ---(6-83)

and hence the degree of freedom can be conveniently incorporated in the t axis by fixing T_o at some arbitrary point. A value of x should then be defined at a starting point where time $t = 0$. This value of x must be determined before a series method of solution can be used.

An alternative series solution suggested by S.C.J. Smith, B.E. (Hons), B. Sc. in terms of sines and cosines appears in the appendix.

Briefly, it is postulated that the $(x_1 t)$ curve cycles harmonically i.e. with no singularities, after infinite time and at this stage an estimate of the purely harmonic part can be made, i.e. as $t \to \infty$

$$x = a_o + \sum_{m=1}^{\infty} (a_m \sin t)^m + \sum_{n=1}^{\infty} (a_n \cos t)^n + \sum_{p\ q=1}^{\infty} (a_p \sin t)^p (a_q \cos t)^q$$

But $(\cos t)^{2m+1} = \cos t \ \ \Sigma\ (\sin^2 t)^m$

and similarly $(\sin t)^p\ (\cos t)^q = \cos t \ \ \Sigma\ (\sin t)^{p+2}$

Hence: $x = (a_o + a_1 \sin t + a_2 \sin^2 t + \text{---------})$

$$+ \cos t\ (b_o + b_1 \sin t + b_2 \sin^2 t + \text{----------})$$

and solutions of a_o, a_1, a_2 -------, b_o, b_1, b_2 ------- must be

obtained to satisfy equation (6-71) viz:-

$$\frac{dx}{dt} = \frac{1}{h} \left(\Phi(x) \sin t + \Psi(x) \right)$$

A method of successive substitution was also tried.

If the n th approximation to x be x_n,

$$x_n = x_o + \int_o^t f(x_{n-1}, t)\, dt$$

and for $x = 0.06$ at $t = 0$

$$x_1 = 0.060 + \int_o^t f(0.06, t)\, dt$$

Where $f(0.06, t)$ is the solution of equation (6-66) for $x = 0.06$ for all t.

It is found that:- $f(0.060,t) = -0.101 - 0.172 \sin t$

$$\therefore \quad x_1 = 0.060 + 0.172 \cos t - 0.101\, t$$

and similarly:- $x_2 = 0.060 + \int_o^t f\left(\left[0.060 + 0.172 \cos t - 0.101t \right], t \right) dt$

Clearly this becomes most complex, even if cos t is replaced by a series in t, and must be repeated for all the arbitrary values of x at $t = 0$. It also appears likely that a considerable number of estimates for each value of x at $t = o$ would be necessary to stabilise the series in t, up to (say) terms in t^4.

Before proceeding, it is necessary to establish some meaning for a_o in these series solutions. No value of x at any given value of t is known for a 2 pond system, so that some logical estimation is required.

It is clear, however, that the (x, t) curve is to be a periodic function since the curve becomes the projection on an (x, t) plane of a skew line on the three dimensional surface of $(x, t, \frac{dx}{dt})$. This surface can be evaluated, since $h\frac{dx}{dt} = \Phi(x) \sin t + \Psi(x)$ and for any given x, $\frac{dx}{dt}$ is sinusoidal with time. For any given t, $\frac{dx}{dt}$ is a function of x of order x^4.

Values of $h \frac{dx}{dt}$ can be obtained for given values of x and t from which the surface can be drawn.

A qualitative picture of this surface is given in Fig. (6.2). From this qualitative picture several points arise:

(i) In practice $0.0199 \leq x \leq 0.137$ and therefore the surface is only valid between these limits of x.

(ii) At $x = 0.137$ $\frac{dx}{dt} = -\infty$

(iii) For $t = 0$ $\frac{dx}{dt} = \Psi(x)$

$= 0$ when

$$0.14536x^2 - 0.16129x + 0.00695 = 0$$

$$\text{i.e. } x = \frac{0.16129 \pm \sqrt{0.026014 - 0.004041}}{0.29072}$$

$$= \frac{0.16129 \pm 0.14823}{0.29072}$$

$$= 1.06467,\ 0.04492$$

Within the practical limits of x, $\frac{dx}{dt} = 0$ when $t = 0$ and $x = 0.04492$.

(iv) When $\frac{dx}{dt} = 0$, the $(x_1 t)$ curve passes through a maximum or minimum and the surface cuts the (x,t) plane.

(v) $\Phi = 0$ when, for values within the practical range:

$$0.08166x^2 - 0.42028x + 0.01536 = 0$$

$$\text{i.e. } x = \frac{0.42028 \pm \sqrt{0.17664 - 0.005017}}{0.16332}$$

$$\text{i.e. } x = 0.03680,\ 5.1099$$

At the point $x = 0.03680$ (within the range of the practical values of x), for all t.

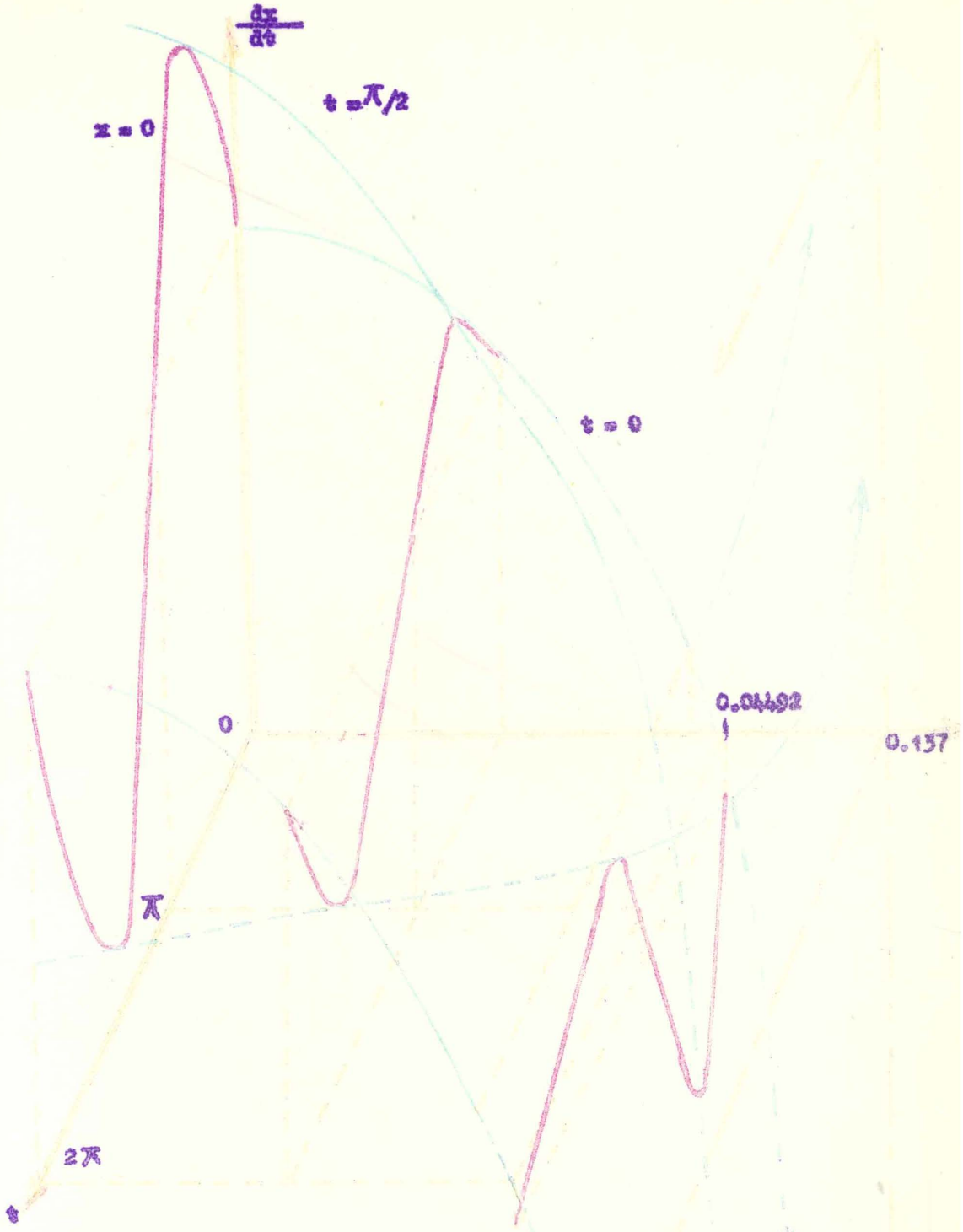
$$\frac{dx}{dt} = \frac{(1-1.27x)^2}{(0.137 - x)} (0.14536x^2 - 0.16129x + 0.00695)$$

$$= +0.011521$$

That is at $x = 0.03680$, $\frac{dx}{dt}$ is a constant for all t.

(vi) For all values of $x > 0.03680$, $\Phi(x)$ is negative and the curve, at any x, of $\frac{dx}{dt}$ versus t begins with a trough rather than a ridge in the range $t = 0$ to π

That is, this curve becomes negative sine curve.

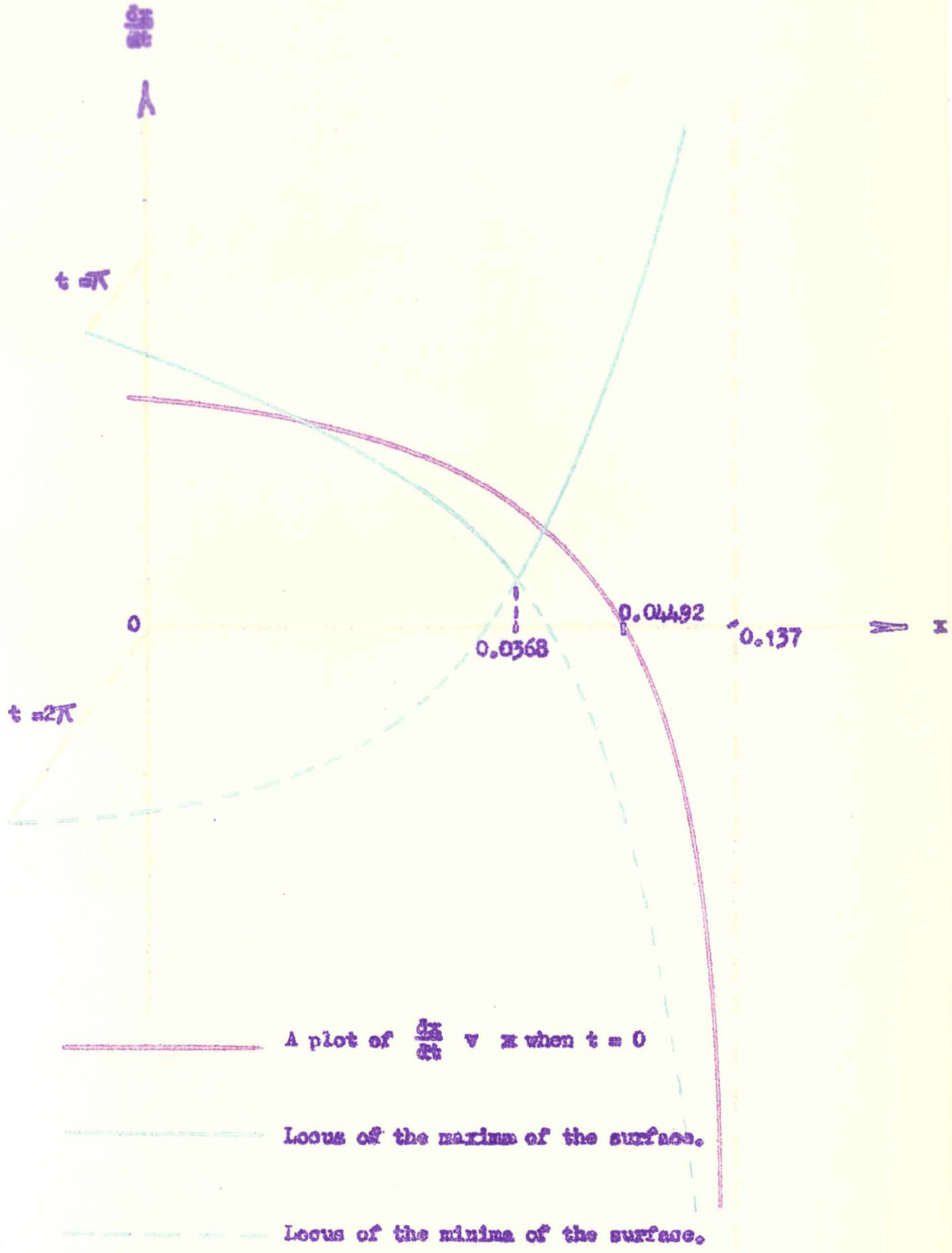


——— $\frac{dx}{dt}$ as a function of t only

$\frac{dx}{dt}$ as afunction of x only when $\frac{dx}{dt}$ is on an apex

$\frac{dx}{dt}$ as a function of x only when $\frac{dx}{dt}$ is in a trough

Fig. (6.2) A qualitative picture of the three dimensional surface of (x, t, $\frac{dx}{dt}$)



A plot of $\frac{dx}{dt}$ v x when t = 0

Locus of the maxima of the surface.

Locus of the minima of the surface.

Note: At x = 0.03680 maxima become minima and vice versa.

Fig. (6.3) A plot of $\frac{dx}{dt}$ v x at t = 0 and the locus of the maxima and minima at t = π and 2π .

From this surface, a qualitative picture of the $(x_1 t)$ curve can be derived.

(i) The limits of x in a system is $0.0199 \leq x \leq 0.137$ so that the curve over the surface which produces the $(x_1 t)$ curve as a projection on the $(x_1 t)$ plane must be a skew line, oscillating somewhere and in some way between the bounds of $0.0199 < x < 0.137$.

(ii) It is clear that $\frac{dx}{dt}$ is a term which has two meanings:-

$\frac{dx}{dt}$ is the height of the surface above or below the $(x_1 t)$ plane.

$\frac{dx}{dt}$ rewritten as $\frac{d}{dt}(x)$ or $\frac{d}{dt}\ f(t)$, because $x = f(t)$, is the slope of the $(x_1 t)$ curve.

Thus (the slope of the $(x_1 t)$ curve at any point)
= (distance of the skew line on the surface from the $(x_1 t)$ plane for that point)

(iii) Because of (ii), as the distance of the surface from the $x_1 t$ plane varies, $\frac{dx}{dt}$ can not remain constant for all values of x and t, so that the $(x_1 t)$ curve will not be a straight line, or tend to a straight line after any time.

i.e. $x = f(t)$ must be a periodic function.

(iv) $x = f(t)$ could be repetitive every cycle of t or differ every cycle, since there is nothing to demand the skew curve being repetitive. The $(x_1 t)$ curve must, however, have the same period as t although it may be anything up to $\frac{\pi}{2}$ out of phase.

(v) It appears likely, that if the $(x_1 t)$ curve is different for every cycle, it will die away to a repetitive function after some time, for if it did not, the $(x_1 t)$ curve would be completely random for all time.

(vi) When the surface cuts the $(x_1 t)$ plane, $\frac{dx}{dt} = 0$ so that a maximum or minimum occurs in the $(x_1 t)$ curve.

When $\frac{dx}{dt}$ = maximum or a minimum, $\frac{d}{dt}\left(\frac{dx}{dt}\right) = 0$, and a point of inflection occurs in the $(x_1 t)$ curve.

From these points it is clear that the $(x_1 t)$ curve must be a periodic function, oscillating somewhere within the practical limits of x. For this to happen $\frac{dx}{dt} = 0$ in this range, so that the $(x_1 t)$ curve may have maximum and minimum values.

Remembering that:-

$$\frac{dx}{dt} = \frac{(1-1.27x)^2}{(0.137-x)} \left[\begin{array}{l} (0.08166x^2 - 0.42028x + 0.01536) \sin t \\ + (0.14536x^2 - 0.16129x + 0.00695) \end{array} \right]$$

$= 0$ for certain values of x and t so that the $(x_1 t)$ curve may have stationary values, the limits of x can be determined.

From equation (6-66), for $\frac{dx}{dt} = 0$,

(i) $(1-1.27x) = 0$ which gives $x = \frac{1}{1.27} = 0.79$

(ii) $(0.137 - x) = 0$ " " $x = 0.137$

(iii) $\Psi(x) = 0$ " " $x = 1.0647,\ 0.04492$

and

(iv) $\Phi(x) = 0$ which gives $x = 5.1099,\ 0.03680$

Conditions (i), (ii) above have no value within this problem so that the conditions (iii) and (iv) must apply (together) i.e.

$$\Phi_1(x) \sin t + \Psi_1(x) = 0$$

i.e. For the $(x_1 t)$ curve to have stationary values in the range $0.0199 \leq x \leq 0.137$,

$\Phi_1(x) \sin t + \Psi_1(x) = 0$ in the same range of x.

Thus $\sin t = -\Psi_1(x) / \Phi_1(x)$

For this to hold since $-1 \leq \sin t \leq +1$, the condition $-1 \leq \frac{\Psi_1(x)}{\Phi_1(x)} \leq +1$ must define the values of x for stationary points on the curve.

CALCULATIONS:

(a) $\Psi_1(x) / \Phi_1(x) = +1$

i.e. $0.08166x^2 - 0.42028x + 0.01536$

$= 0.14536x^2 - 0.16129x + 0.00695$

i.e. $0.06370x^2 + 0.25899x - 0.00841 = 0$

i.e. $x = 0.03226, -4.0980$

(b) $\Psi_1(x) \Phi_1(x) = -1$

i.e. $0.22702x^2 - 0.58157x + 0.02231 = 0$

i.e. $x = +2.5228, 0.03895$

(c) $\Psi_1(x) / \Phi_1(x) = 0$ when $\Psi_1(x) = 0$

i.e. $x = +1.0647, 0.04492$

(d) When $x \to \pm \infty$

$$\Psi_1(x) / \Phi_1(x) \to \frac{0.14536}{0.08166}$$

$$\to 1.7801$$

(e) $\Psi_1(x) / \Phi_1(x) \to \infty$ when $\Phi_1(x) \to 0$

i.e. when $x \to 5.1099, 0.03680$

Note however that:

for $x > +1.0647$, $\Psi_1(x)$ is positive

and for $x < +0.04492$, $\Psi_1(x)$ is positive

Similarly for $0.03680 < x < 5.1099$, $\Phi_1(x)$ is negative.

Hence: $\Psi_1(x) / \Phi_1(x) \to +\infty$ as

$x \to +0.03680$ in a positive direction

and $\to -\infty$ as $x \to 0.03680$ in a negative direction.

and $\Psi_1(x) / \Phi_1(x) \to -\infty$

as $x \to 5.1099$ in a positive direction

and $\to +\infty$ as $x \to 5.1099$ in a negative direction.

From these factors, a qualitative picture of the limits of the (x,t) curve can be derived.

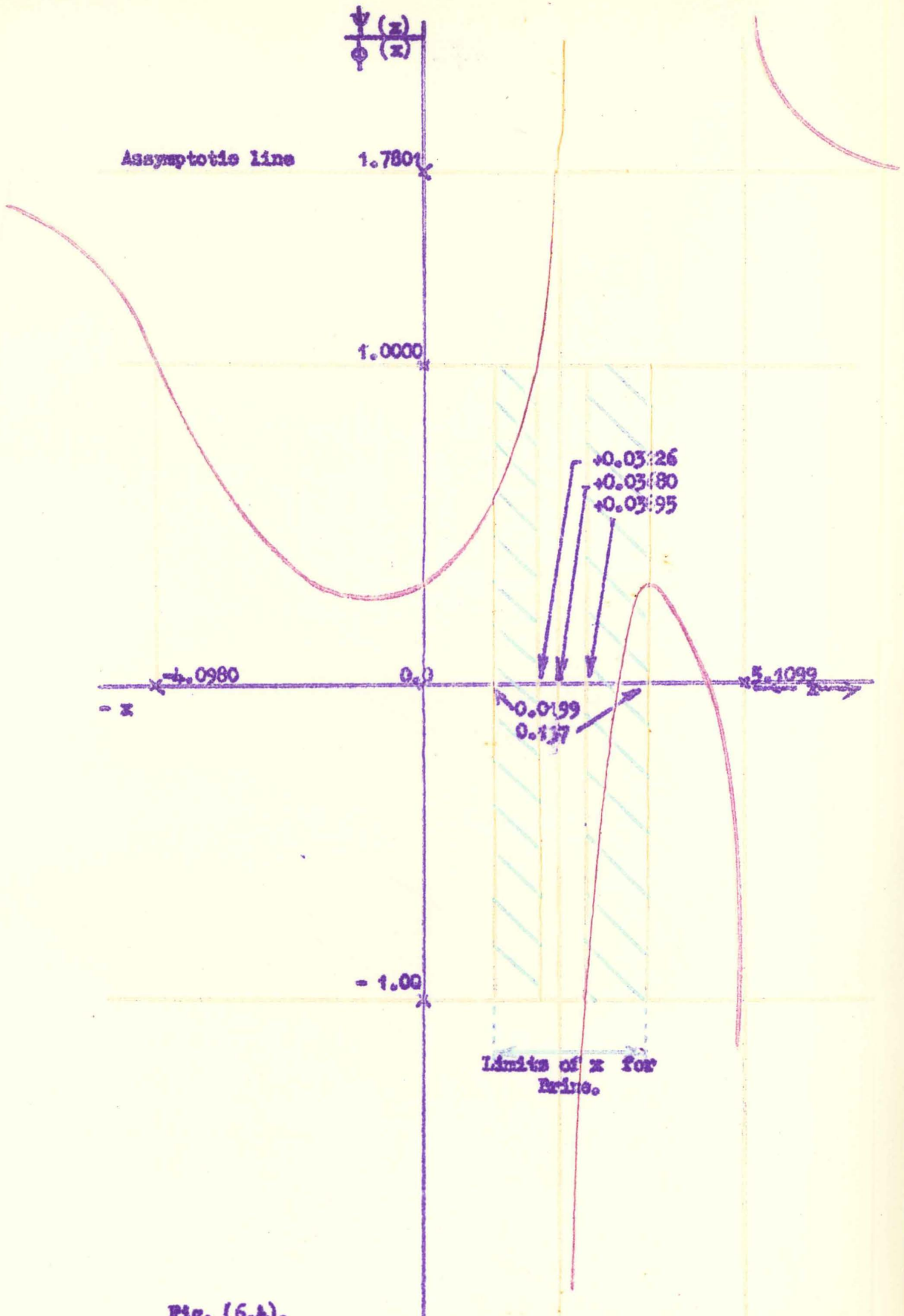


Fig. (6.4).

A semi-quantitative picture of the plot of $\psi(x)/\phi(x)$ for various values of x.

The curve of $\Psi_1(x)/\Phi_1(x)$ is shown in figure (6.4)

From this figure:

(i) The curve of $\Psi_1(x)/\Phi_1(x)$ lies within the bounds

$-1 \leq \Psi_1(x)/\Phi_1(x) \leq +1$ only for values of x within the limits of:

$$\left.\begin{aligned} -4.0980 \leq x \leq +0.03226 \\ \text{and } +0.03895 \leq x \leq +2.5228 \end{aligned}\right\} \quad \text{----- (6-84)}$$

Hence only values of x within the ranges defined by equation (6-84) are real, if there are to be any stationary values in the (x,t) curve.

(ii) For the sea water problem, values of x must be in the range

$$+0.0199 \leq x \leq 0.137$$

so that the actual working values of x, if there are to be any stationary values in the (x,t) curve must lie in the limits defined by:

$$\left.\begin{aligned} +0.0199 \leq x \leq +0.03226 \\ \underline{\text{or}} \quad +0.03895 \leq x \leq 0.137 \end{aligned}\right\} \text{----(6-85)}$$

(iii) Equation (6-85) above, implies that only when pond No. 1 is operating with concentration x defined within these limits, is there any possibility of a maximum, a minimum or a point of inflection occurring in the (x,t) curve and hence any possibility of cycling or periodic fluctuation of x with time.

Should the pond concentration exceed these limits for some reason, at some particular time, then x will vary with time in some steadily increasing or decreasing manner until it again enters the limits defined in equation (6-85)

Since the range of stability defined by:

$0.0199 \leq x \leq 0.03226$ is very narrow, it suggests that unless control can be assured in some way to prevent momentary fluctuations outside the limits (such as would be caused by sudden rainfall or a prolonged spell of hot, dry weather), pond number one should be operated within the limits of stability defined by:- $0.03895 \leq x \leq 0.137$ ----- (6-86)

This derivation of ranges of values in which x can operate is borne out by the solution of the equation as shown in figures (6.6 to 6.11). In the range of 0.03226 to 0.03895, no oscillatory curve can be found for any of the curves, with x increasing with time until the upper value is reached when cycling can continue.

When x has values in the lower range, it tends to increase into the upper range and the final oscillatory curves are in this range $0.03895 \leq x \leq 0.137$. It is evident that this is the most stable range of values of x for pond one.

If the (x,t) curve of operation in this range $0.03895 < x < 0.137$ approaches at any time either of the limiting boundaries of

$$x = 0.03895 \quad \text{or} \quad x = 0.137,$$

a period of potential instability would result, since, by exceeding these limits, rapid dilution (into the other range of stability) or crystallisation out of salt would occur.

This, then, has defined the limits for x, and since the degree of freedom of equation (6-71) can be incorporated into the time axis, it is clear that, providing the value of x for pond one is within the defined limits, x may have any value at time $t = 0$. This means a_o may have any value defined by these limits. For this reason the series solutions, involving much arithmetic computation were left, and a method of numerical analysis investigated, in which, various values of x at time $t = 0$ were used to establish families of curves of x versus t, at any given h. This process was intended to be carried out for a number of time periods to see if all the curves remained in families or if they all tended to one curve at some time. It is difficult to determine this from the form of equation (6-66) but clearly it is important.

To help in this analysis, the locus of the stationary values (defined by $\frac{dx}{dt} = 0$) and of the points of inflection (defined by $\frac{d^2x}{dt^2} = 0$) were plotted.

For these : $$\frac{dx}{dt} = \frac{1}{h} \frac{(1-1.27x)^2}{0.137 - x} \left[\Phi_1(x) \sin t + \Psi_1(x) \right] \qquad \text{----(6-71)}$$

when $\sin t = 0$, $\Psi_1(x)$

$= -\Psi_1(x)/\Phi_1(x)$

since in the defined limits of x,

$$(1-1.27x) \neq 0$$

$$\text{or } (0.137 - x) \neq \pm \cdots$$

and since h is assumed $\neq \pm \cdots$

The values of t, at given values of x, are tabulated in table (6.2) and plotted in figure (6.5).

For the points of inflection, differentiating equation (6-71) and equating to zero gives:-

$$\frac{(1-1.27x)^2}{(0.137-x)}\left[\Phi_1(x)\sin t + \Psi_1(x)\right]\left\{\left[\Phi_1(x)\sin t + \Psi_1(x)\right]\frac{0.65202 + 1.27x}{0.137 - x} + (1-1.27x)\left[\frac{d\Phi}{dx}1 \sin t + \frac{d\Psi}{dx}1\right]\right\} + \Phi_1(x)\cos t = 0 \qquad \text{---- (6-87)}$$

where: $\Phi_1(x) = 0.08166x^2 - 0.42028x + 0.01536$

$$\Psi_1(x) = 0.14536x^2 - 0.16129x + 0.00695$$

$$\frac{d\,\Phi_1(x)}{dx} = 0.16332x - 0.42028$$

$$\frac{d\,\Psi_1(x)}{dx} = 0.29072x - 0.16129$$

Inserting values of x in these expressions, gives equations of the form:

$$A\cos t + B\sin^2 t + C\sin t + D = 0 \qquad \text{---- (6-88)}$$

which was solved by a trial and error procedure to find values of t, given in table (6.3) and figure (6.5).

Where $A = 1.0$

$$B = -\frac{(1-1.27x)}{(0.137-x)}\left[(1-1.27x)\frac{d\Phi_1}{dx} + \left(\frac{1.27x + 0.65202}{0.137 - x}\right)\Phi_1(x)\right]$$

$$C = -\frac{1}{\Phi_1(x)}\left\{\frac{(1-1.27x)^2}{(0.137-x)}\left[\Phi_1(x)\frac{d\Psi_1}{dx} + \Psi_1(x)\frac{d\,\Phi_1}{dx}\right] + \Phi_1(x)\cdot\Psi(x)\frac{2(1-1.27x)}{(0.137-x)}\cdot\frac{(1.27x + 0.65202)}{(0.137 - x)}\right\}$$

$$D = \frac{-1}{\Phi_1(x)}\cdot\frac{1-1.27x}{0.137-x}\cdot\Psi_1(x)\left[(1-1127x)\frac{d\Psi_1}{dx} + \frac{1.27x+0.65202}{(0.137 - x)}\Psi_1(x)\right]$$

x	sin t	t	
0.039	+0.9713	76°14'	103°46'
0.0399	+0.5833	35°41'	144°19'
0.0394	+0.7612	49°34'	130°26'
0.0392	+0.8596	60°16'	119°44'
0.0397	+0.6488	40°27'	139°33'
0.0409	+0.3520	20°37'	159°23'
0.0415	+0.2612	15°9'	164°51'
0.0426	+0.1432	8°14'	171°46'
0.03895	+1.0000	90°	180°
0.03905	+0.9417	70°20'	109°40'
0.044	+0.0454	+ 2°36'	177°34'
0.047	-0.0736	- 4°13'	184°13'
0.050	-0.1378	- 7°55'	187°55'
0.053	-0.1779	- 10°15'	190°15'
0.058	-0.2192	- 12°40'	192°40'
0.060	-0.2311	- 13°22'	193°22'
0.065	-0.2514	- 14°34'	194°34'
0.067	-0.2578	- 14°56'	194°56'
0.070	-0.2656	- 15°24'	195°24'
0.073	-0.2721	- 15°47'	195°47'
0.075	-0.2757	- 16°0'	196°0'
0.077	-0.2789	- 16°12'	196°12'
0.085	-0.2887	- 16°47'	196°47'
0.087	-0.2906	- 16°54'	196°54'
0.089	-0.2922	- 16°59'	196°59'
0.090	-0.2930	- 17°2'	197°2'
0.093	-0.2951	- 17°10'	197°10'
0.095	-0.2963	- 17°14'	197°14'
0.055	-0.1973	- 11°23'	191°21'

Table (6.2) - Values of t and x at which $\frac{dx}{dt} = 0$

x	Quadrant 1		2		3		4	
	sin t	t	sin t	t	sin t	t	sin t	t
0.04	.71962	$46^\circ 2'$	—	—	—	—	-0.53666	$-32^\circ 27'$
0.05	.23272	$13^\circ 28'$	—	—	—	—	-.65166	$-40^\circ 41'$
0.07	.081465	$4^\circ 40\frac{1}{2}'$	—	—	—	—	-.62592	$-38^\circ 45'$
0.08	0.02531	$1^\circ 27'$	—	—	—	—	-.59244	$323^\circ 40'$
0.10	—	—	—	—	—	—	-.08739 -0.50636	$354^\circ 59'$ $329^\circ 35'$
0.11	—	—	—	—	—	—	-0.4563 -0.17479	$-27^\circ 9'$ $-10^\circ 4'$

Table (6.3) - Points of inflection in the $x_1 t$ curve.

(Where a position in this table has a line through it, it has been proved that no value of t exists for that x for which $\frac{d^2x}{dt^2} = 0$.)

From plots of the loci of $\frac{dx}{dt} = 0$ and $\frac{d^2x}{dt^2} = 0$, Figure (6.5)

it is seen that the $\frac{dx}{dt} = 0$ curve is enveloped in the fourth quadrant by the $\frac{d^2x}{dt^2} = 0$ curve. These cross only if $\frac{d^2x}{dt^2} = \frac{dx}{dt} = 0$, or

$$\frac{(1-1.27x)^2}{(0.137-x)}\left[\Phi_1(x)\sin t + \Psi_1(x)\right]$$

$$= \frac{(1-1.27x)}{(0.137-x)}\left[\Phi_1(x)\sin t + \Psi_1(x)\right]\left\{\left[\Phi_1(x)\sin t + \Psi_1(x)\right]\frac{0.65202+1.27x}{(0.137-x)}\right.$$

$$\left. + (1-1.27)\left[\frac{d\Phi_1}{dx}\sin t + \frac{d\Psi_1}{dx}\right]\right\} + \Phi_1(x)\cos t = 0$$

But $x \neq \frac{1}{1.27}$ or 0.137 and hence the only solution is for:-

$$\Phi_1(x)\cos t = 0 \text{ and } \left[\Phi_1(x)\sin t + \Psi_1(x)\right] = 0$$

and $\therefore \cos t = 0 \quad \because \Phi_1(x)$ & $\Psi_1(x) \neq 0$ simultaneously,

i.e. $\Psi_1(x) / \Phi_1(x) = \pm 1$

and hence $x = 0.03226,\ 0.03895 \quad (-4.0980,\ +2.5228)$

Hence these curves do not meet above $x = 0.03895$ in the practical range of x.

These facts gave a qualitative picture of the (x, t) curve, and a numerical analysis of equation (6-66) was attempted. The method of Picard (Piagio. "Differential equations, page 94), was used with successive approximation until a solution was obtained which was correct in the sixth decimal place.

An arbitrary starting value of $x = 0.060000$ at $t = 0.0$ $\left(\frac{dx}{dt} = \frac{-0.024427}{h}\right)$ was tried firstly with $h = 1"$ and then $h = 12"$. This process was a very time consuming one particularly as the process appeared inadequate at points of inflection. This meant that the point of inflection itself had to be determined by a solution of $\frac{d^2x}{dt^2} = 0$ simultaneously with a solution satisfying the previous (x, t) curve.

By this method the results of tables (6.4 and 6.5) were calculated and plotted on figure (6.5).

From these plots, it is clear that $h = 1"$ is too shallow a depth to produce cycling of the (x, t) curve, for, as t increases, x begins to cycle and then plunges with infinite slope through the boundary of $x = 0.137$. That is, salt would crystallise out and no saturated brine would be produced from the system.

The 12" depth solution shows that the value of x at $t = 2\pi$ does not equal the value of x at $t = 0$ so that further periods would be necessary to determine the final, purely oscillatory, part of the curve of x versus t.

Table (6.4) - Points on x = f(t) curve.

TWO POND SYSTEM.

h = 1 inch depth

Assume x = 0.06 when t = 0.0000		
t	x	Slope $\frac{dx}{dt}$
0	0.06000	-0.02438
$\frac{\pi}{6}$	0.04720_5	-0.02447_2
$\frac{\pi}{3}$	0.03990_8	-0.00340_7
$\frac{\pi}{2}$	0.03920_9	-0.00133_9
$\frac{2\pi}{3}$	0.03904_1	$+0.00070_1$
$\frac{5\pi}{6}$	0.03967_6	$+0.00172_3$
π	0.04140_7	$+0.00489_2$
$\frac{\pi}{6}$	0.04737_8	$+0.01792_7$
$\frac{15\pi}{12}$	0.05501_8	$+0.04042_8$
230°	0.05917_0	$+0.05471_9$
235°	0.06490_6	0.07678_5
$4\frac{\pi}{3}$	0.07326_5	0.11482_2

Table (6.5) - Points on x = f(t) curve.

TWO POND SYSTEM.

h = 12" depth.

t	x	Slope $\frac{dx}{dt}$
0	assume 0.06	- 0.002036
$9^{\circ}21'$	Point of inflection 0.05975	- 0.003413
0	0.060096	-
$\frac{\pi}{6}$	0.05810_2	- 0.005728
$\frac{\pi}{3}$	0.05479_3	- 0.006914
$\frac{\pi}{2}$	0.05142_6	- 0.005950
$\frac{2\pi}{3}$	0.04880_4	- 0.004007
$\frac{5\pi}{6}$	0.04721_5	$- 0.00204_4$
π	0.04662_5	- 0.000207
$\frac{7\pi}{6}$	0.04695_4	+ 0.001468
$\frac{4\pi}{3}$	0.048111	$+ 0.00295_3$
$\frac{3\pi}{2}$	0.04990_9	$+ 0.00392_3$
$\frac{5\pi}{3}$	0.05191_4	$+ 0.00373_4$
320°	0.05361_4	$+ 0.00276_0$
340°	0.05458_6	$+ 0.00095_4$

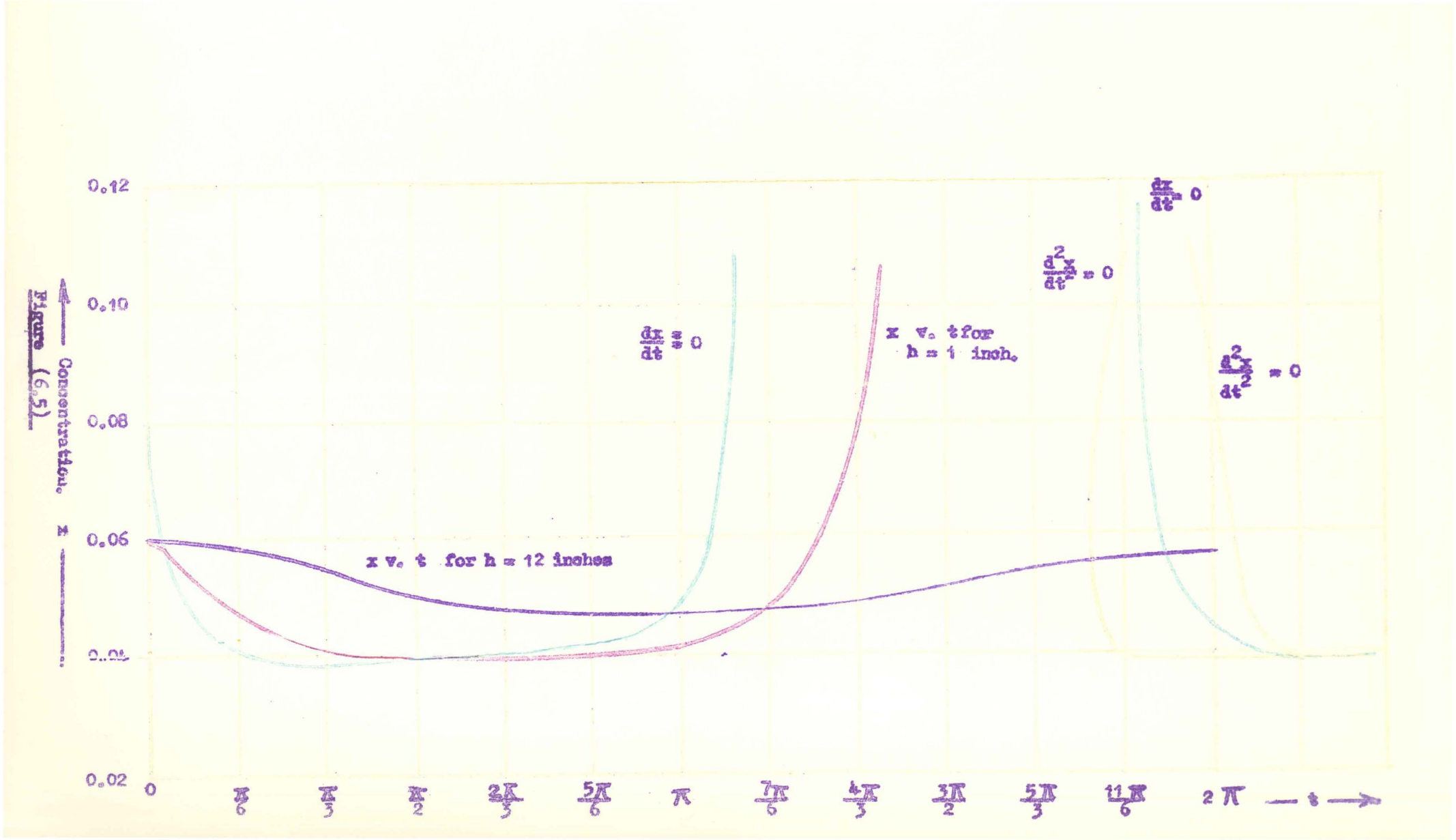

0.12
0.10
0.08
0.06
0.04
0.02
0
π/6
π/3
π/2
2π/3
5π/6
π
7π/6
4π/3
3π/2
5π/3
11π/6
2π
— t —
Concentration, x
dx/dt = 0
d²x/dt² = 0
dx/dt = 0
d²x/dt² = 0
x v. t for h = 1 inch.
x v. t for h = 12 inches

Figure (6.5)

At this stage, Dr. R.W. Williams, and his assistants of the Department of Applied Mathematics in D.S.I.R. offered to carry out a numerical analysis of equation (6-66) on the differential analyser, and solutions were attempted for

$$h = 3", 6", 12", 18", 36"$$

and at each value of h, the various starting values of x at $t = 0$ of

$$0.02, 0.03, 0.04, 0.05, 0.06, 0.08.$$

The solutions were produced in the form of curves from $t = 0$ to $t = 6\pi$ with $6" \equiv 2\pi$ on the time axis and $1" \equiv 0.005$ on the concentration axis. For these solutions values of $\Phi(x)$ and $\Psi(x)$ were provided as in table (6.6).

The solutions for all these conditions are given in figures (6.6),(6.7),(6.8),(6.9),(6.10) and (6.11). The latter is the solutions of equation (6-71) for different values of h after a very long period of time, when the variation of x with t is reduced to a purely harmonic oscillation regardless of the value of x when $t = 0$. It is clear from all the solutions that such purely harmonic solutions exist which are functions of the pond depth only. This means that under average weather conditions, the variation of concentration with time will tend to a purely periodic function after a long period (see Append).

The solution for $h = 3"$ is probably in error since it was found that the solution could not be repeated. The maxima of this curve falls on the very steep part of the curve of x v t for $\frac{dx}{dt} = 0.$ so that the maxima depends very critically on the values of x and t at the minima, a small error in the minimum leads to a large error in the maximum.

For $h = 6"$ and above no such difficulties arose and the estimated error is of the order of $2\frac{1}{2}\%$. Thus, the uncertainty in (say) $x = 0.04$ is

$$\pm \frac{2.5}{100} \times 0.04 = \pm 0.001$$

It is clear from these curves that the greater the brine depth, the more stable is the system with less vigorous oscillations in concentration. The fact that all the curves tend to a single harmonic curve at all depths, but a different curve for each depth, eliminates the necessity for evaluating a_o and for this reason the

Table of $\Phi(x)$ and $\Psi(x)$ for different values of x for the equation

$$\frac{dx}{dt} = \Phi(x) \sin t + \Psi(x)$$

x	$\Phi(x)$	$\Psi(x)$
0.020	+0.056723	+0.030704
0.023	+0.047428	+0.027421
0.025	+0.041049	+0.025187
0.027	+0.034523	+0.022899
0.030	+0.024428	+0.019430
0.032	+0.017487	+0.016987
0.035	+0.006714	+0.013275
0.037	-0.000718	+0.010726
0.039	-0.008361	+0.008121
0.0426	-0.022709	+0.003251
0.044	-0.028504	+0.001294
0.047	-0.041390	-0.003046
0.048	-0.045824	-0.004528
0.049	-0.050346	-0.006036
0.050	-0.054941	-0.007571
0.051	-0.059619	-0.009133
0.052	-0.064385	-0.010714
0.053	-0.069282	-0.012324
0.055	-0.079206	-0.015626
0.058	-0.094956	-0,020812
0.059	-0.100409	-0.022601
0.060	-0.105689	-0.024427
0.065	-0.135762	-0.34136
0.067	-0.148638	-0.038313
0.070	-0.169230	-0.044949
0.0716	-0.180969	-0.048716
0.073	-0.191453	-0.052092
0.075	-0.207310	-0.057155
0.077	-0.224076	-0.062636
0.080	-0.251199	-0.071126
0.083	-0.281006	-0.080563
0.085	-0.302600	-0.087365
0.087	-0.325764	-0.094663
0.089	-0.350711	-0.102486
0.090	-0.363925	-0.106637
0.093	-0.406894	-0.120071
0.095	-0.438731	-0.129981
0.097	-0.473559	-0.140824
0.100	-0.532481	-0.159120

Table (6.6)

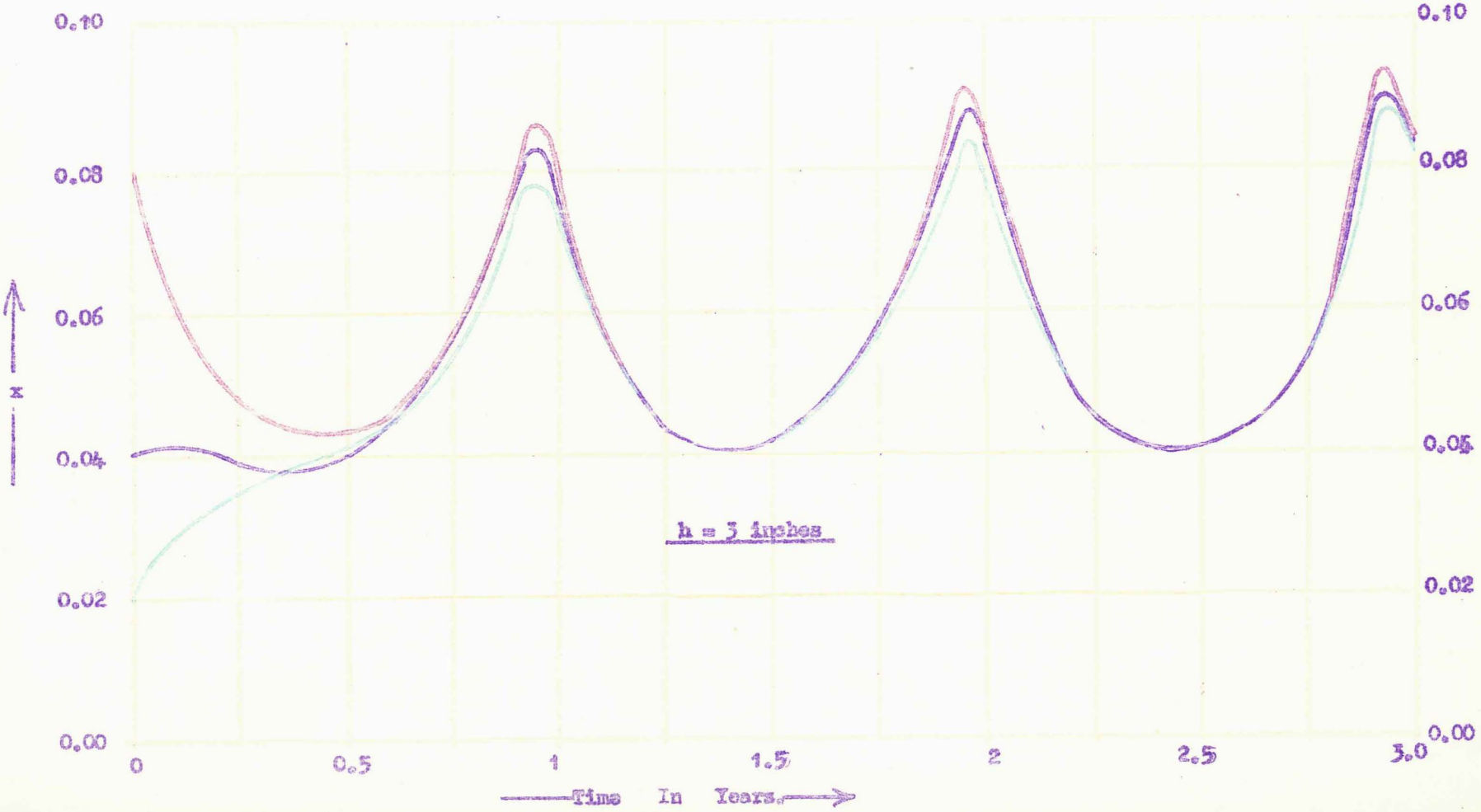
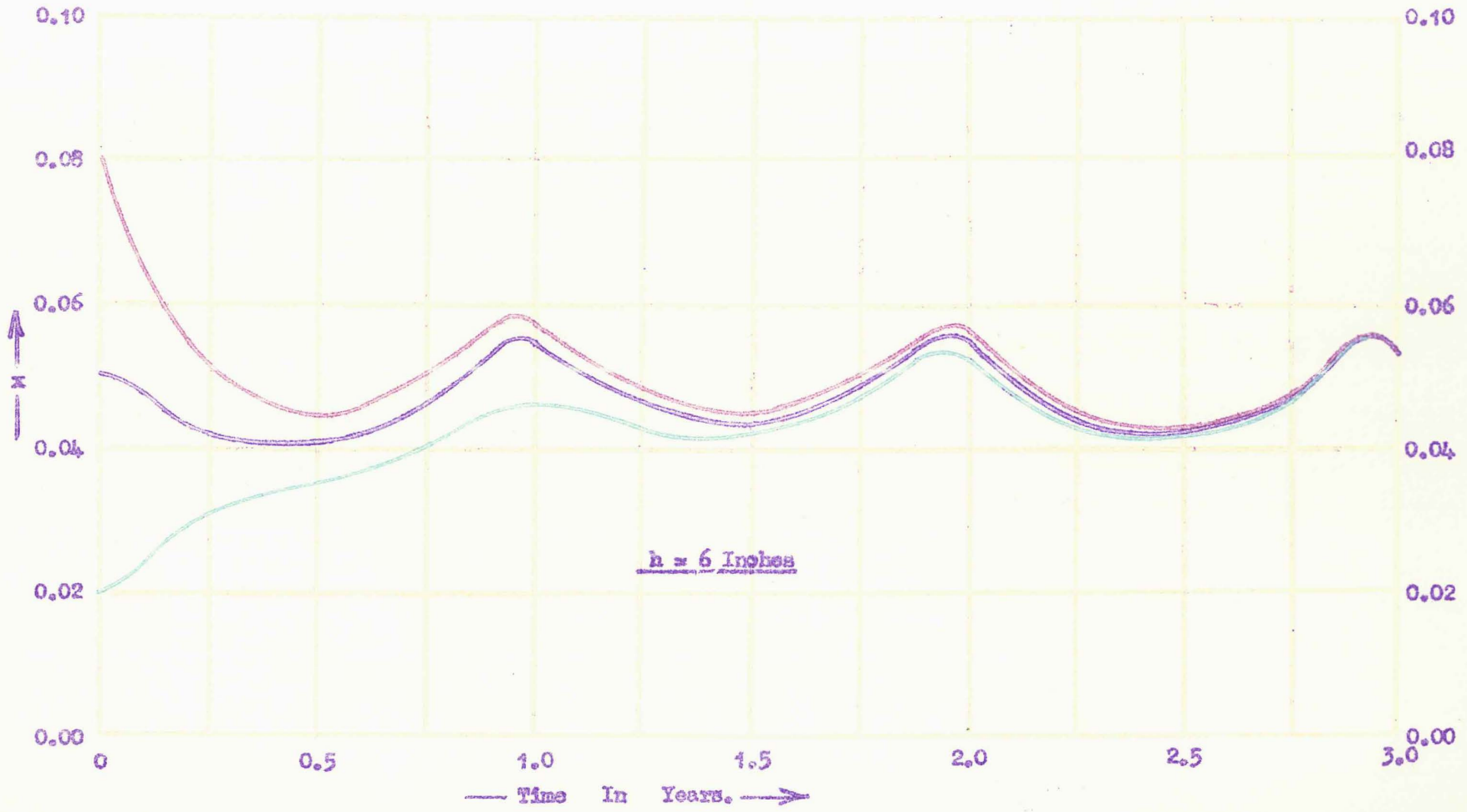


Figure (6.6)

A plot of z versus t for the first three years of operation at different starting values at t = 0.

Fig. (6.7)

Plot of κ versus t



Plot of x versus t for pond one in a two pond system.

$\frac{dx}{dt}$ (ft/yr)

x

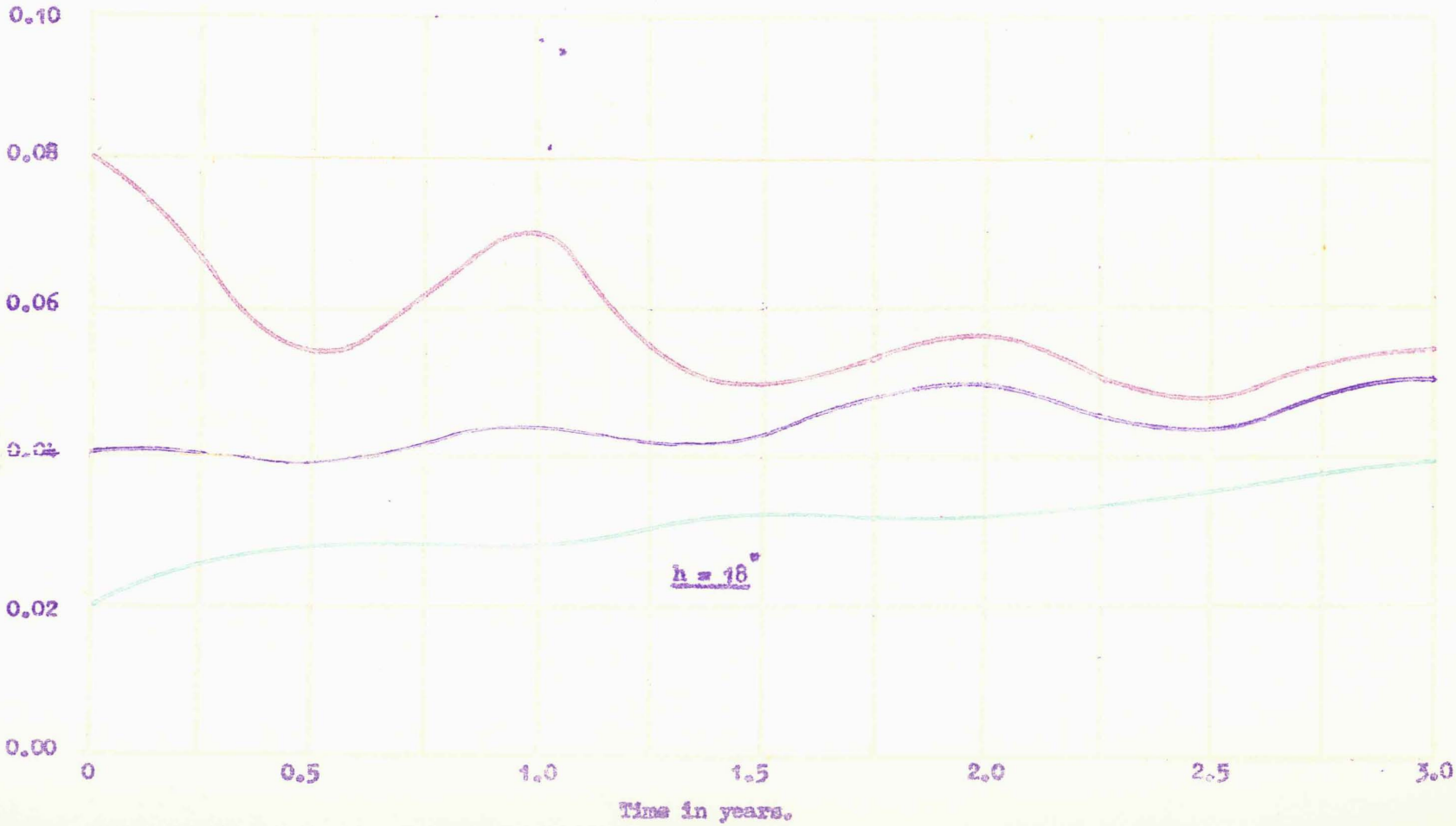
0.10
0.08
0.06
0.04
0.02
0.00

h = 12"

0
0.5
1.0
1.5
2.0
2.5
3.0

Time in years.

Fig. 6.9 Plots of x versus t for point one in a two point system.



A plot of x versus t for point one in a two point system.

Fig. 5.10

Figure (6.11) Plot of x versus t for pond one in a two pond system after a long period of time has elapsed.

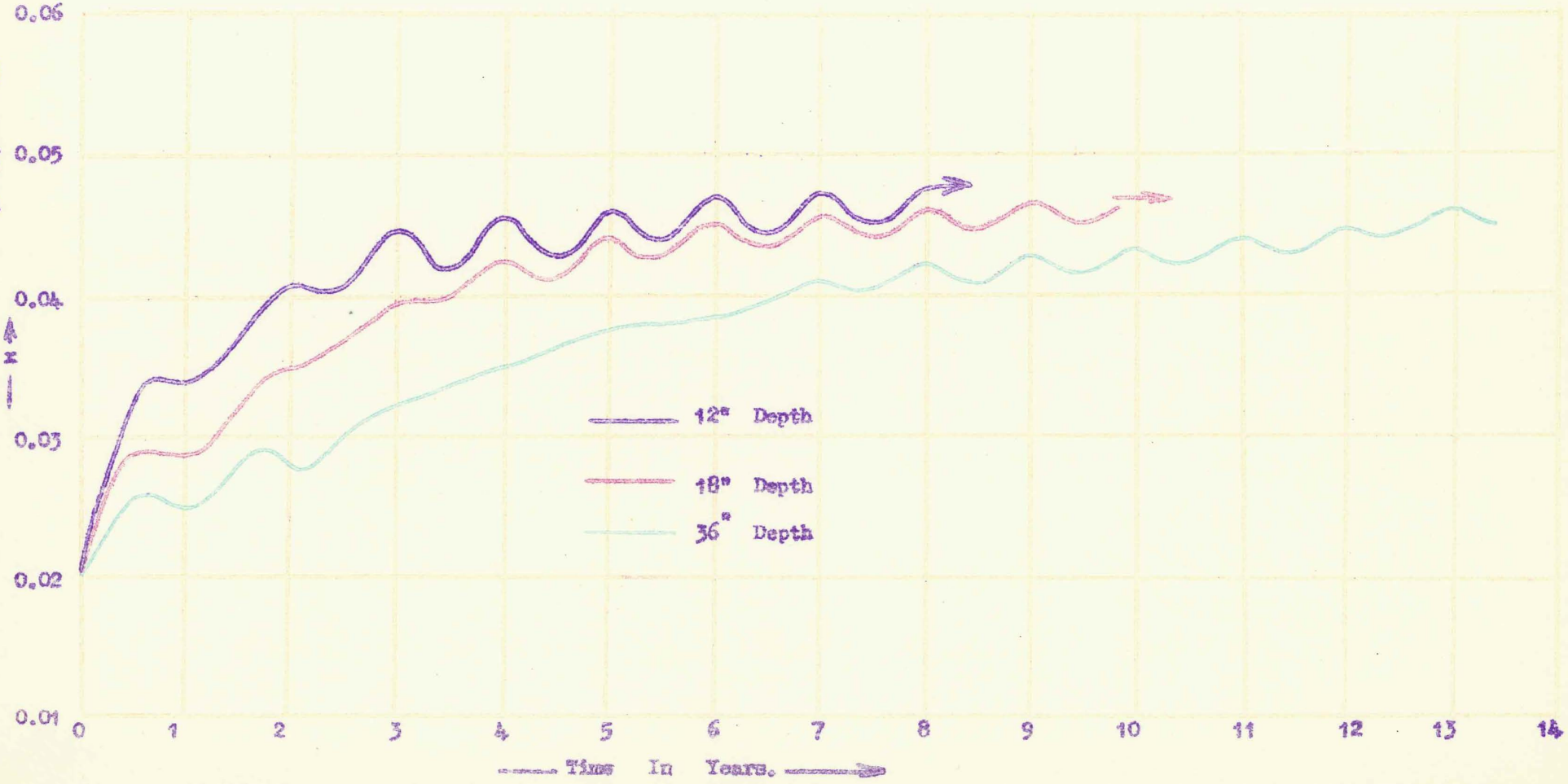


Figure (6.12)
Plot of x v.s. for a 2 pond system for the first nine years of operation.

solutions for all the starting values of x are important. However, it is the oscillatory curves which will give the output of brine which could ultimately be expected from the system.

It is considered that depths of 3" and 6" are impractical so that only the outputs for systems in which h = 12", 18", 36" will be calculated. For these, the values of x at various values of t were extracted as shown in table (6.7)

	h = 12"	h = 18"	h = 36"
t	x	x	x
0	0.0485	0.0471	0.0458
$\frac{\pi}{6}$	0.0477	0.0467	0.0456
$\frac{\pi}{3}$	0.0463	0.0458	0.0452
$\frac{\pi}{2}$	0.0448	0.0448	0.0448
$\frac{2\pi}{3}$	0.0436	0.0440	0.0443
$\frac{5\pi}{6}$	0.0429	0.0435	0.0441
π	0.0428	0.0433	0.0440
$\frac{7\pi}{6}$	0.0432	0.0436	0.0441
$\frac{4\pi}{3}$	0.0442	0.0443	0.0444
$\frac{3\pi}{2}$	0.0455	0.0452	0.0449
$\frac{5\pi}{3}$	0.0471	0.0462	0.0454
$\frac{11\pi}{6}$	0.0482	0.0469	0.0457
2π	0.0485	0.0471	0.0458

Table (6.7) - Values of x at various values of t after a very long period of time. Extracted from Figure (6.11)

From the form of the curves, starting at sea water ($x = 0.0199$). the time required for the system to be purely oscillatory depends on the depth. This would be expected, since the residence time is a direct function of the depth of brine in the pond.

For a three inch depth, it can be seen from figure (6.6) that between 1 and 2 years is all that is required, whereas for 6" about 4 to 5 years and for 12", 18" and 36" it is estimated at 7 to 8 years, 12 years and about 20 years, respectively.

It is important to determine how many years would be required to establish the pure oscillations and just how much saturated brine could be expected from the system in that period if average weather conditions prevailed, since this will determine the economy of the process. Consequently, further solutions of equation (6-71) were obtained from the analyser for the depths h = 12", 18" and 36", starting at $x = 0.02$ at $t = 0$ and continuing each until the purely oscillatory curve was reached. These solutions are given on figure (6.12) and for the 12" depth the values extracted for table (6.8)

Before using these values to calculate yields an investigation was made to see if any of the variables could be so fixed as to give the maximum output for the system. It is clear that regardless of the starting value of x (at $t = 0$), x eventually becomes periodic with t, its value depending on h.

Given: $$F_2 = \frac{A_2 E_2}{x_2 - x_1} \cdot x_1(1-k_1 x_2) \qquad \text{------ (6-24)}$$

where $E_2 = f(x_2, t)$,

the condition of maximum output is obtained by using Euler's theorem of calculus variation -

viz: $$\frac{\partial F_2}{\partial x_1} = \frac{d}{dt}\left(\frac{\partial F_2}{\partial \dot{x}_1}\right) \qquad \text{------ (6-88)}$$

where $\dot{x}_1 = \frac{dx_1}{dt}$

In (6-24) $$F_2 = A_2 E_2 \left[\frac{x_2(1-k_1 x_1)}{(x_2 - x_1)} - 1\right]$$

$$= A_2 E_2 \left[\frac{x_2 h}{(1-kx_1)\ \Phi_1(x) \sin t + \Psi_1(x)} \frac{dx}{dt} - 1\right]$$

From equation (6-71).

Let x_1 = a variable x

and $\Delta = \Phi_1(x) \sin t + \Psi_1(x)$ --- (6-89)

$$\therefore \dot{\Delta} = \frac{d\Delta}{dt} = \left[\frac{d\,\Phi_1(x)}{dx} \sin t + \frac{d\,\Psi_1(x)}{dx} \right] \frac{dx}{dt} + \Phi_1(x) \cos t$$

Then $$\frac{\partial F_2}{\partial \dot{x}} = \frac{A_2 x_2 E_2 h}{(1-kx)\,\Delta}$$ --- (6-90)

$$\therefore \frac{d}{dt}\left(\frac{\partial F_2}{\partial \dot{x}}\right) = + A_2 x_2 \, h \frac{d}{dt} \left[(1-kx)\Delta \right]^{-1} E_2$$

$$= - A_2 x_2 E_2 h \left[(1-kx)\,\Delta \right]^{-2} \left[(1-kx)\dot{\Delta} - k\,\Delta \frac{dx}{dt} \right]$$

$$+ A_2 x_2 h \left[(1-kx)\,\Delta \right]^{-1} \frac{dE_2}{dt}$$ --- (6-91)

Further from equation (6-24) -

$$\frac{\partial F_2}{\partial x} = A_2 E_2 (1-k_1 x_2) \frac{\partial}{\partial x} \cdot \frac{(x)}{(x_2 - x)}$$

$$= A_2 E_2 (1-k_1 x_2) \cdot \frac{(x_2 - x) + x}{(x_2 - x)^2}$$

$$= \frac{A_2 E_2 (1-kx_2)\, x_2}{(x_2 - x)^2}$$ --- (6-92)

$\therefore$ Applying Euler for the maximum and minimum flows from equations (6-88), (6-91) and (6-92).

$$\frac{-A_2 x_2 E_2 h \left[(1-kx)\dot{\Delta} - k\,\Delta \frac{dx}{dt} \right]}{\left[(1-kx)\,\Delta \right]^2} + \frac{A_2 x_2 h}{(1-kx)\Delta} \cdot \frac{dE_2}{dt} = \frac{A_2 E_2 (1-kx_2)\, x_2}{(x_2 - x)^2}$$

and since $A_2 \neq 0$, $x_2 \neq 0$

$$\text{i.e. } - \frac{E_2 h \left[(1-kx)\Delta - k\,\Delta \frac{dx}{dt} \right] (x_2-x)^2}{\left[(1-kx)\,\Delta \right]^2 (x_2 - x)^2} + \frac{h\,(1-kx)\,\Delta}{\left[(1-kx)\,\Delta \right]^2} \cdot \frac{dE_2}{dt}$$

$$- \frac{E_2 (1-kx_2)\,(1-kx)^2\,\Delta^2}{(x_2-x)^2\,(1-kx)^2\,\Delta^2} = 0$$

$$- E_2 h \left[(1-kx)\dot{\Delta} - k\Delta \frac{1}{h} \left(\frac{1-kx}{x_2-x} \right)^2 \Delta \right] (x_2-x)^2 + \frac{h(1-kx)\,\Delta}{1} (x_2-x)^2 \frac{dE}{dt}$$

$$- E_2 (1-kx_2) \left[(1-kx)\Delta \right]^2 = 0$$ --- (6-93)

This equation has several possible roots:

(i) $(1-kx) = 0$

i.e. $x = \frac{1}{k} = 0.787$

This value of x is not in the allowable limit of:-

$$0.0199 \leq x \leq 0.137$$

Then $- E_2 h(x_2-x)^2 \dot{\Delta} + E_2 k \Delta^2 (1-kx) (x_2-x)$

$$+h \Delta(x_2-x)^2 \frac{dE_2}{dt} - E_2 \Delta^2 (1-kx_2) (1-kx) = 0$$

(ii) $x = x_2 = \frac{1}{k} = 0.787$ (twice)

which is not allowable

Then $- E_2 h\dot{\Delta} + E_2 k \Delta^2 + h\Delta \frac{dE_2}{dt} - E_2 \Delta^2 = 0$

(iii) $\Delta = \dot{\Delta} = 0$ gives a solution

For $\Delta = 0$, $\sin t = - \Psi_1 / \Phi_1$

which can hold in the limits defined by :-

$$\left.\begin{array}{l} 0.0199 \leq x \leq 0.03226 \\ \text{or} \\ 0.03895 \leq x \leq 0.137 \end{array}\right\} \text{ from equation (6-85)}$$

With this, $\frac{dx}{dt} = 0$ and hence for $\dot{\Delta} = 0$

$$\Phi_1(x) \cos t = 0$$

i.e. $\Phi_1(x) \sqrt{1-\sin^2 t} = 0$ (for $\Delta = 0$)

i.e. $\sqrt{\Phi^2 - \Psi_1^2} = 0$

or $\Phi_1 = \pm \Psi_1$

From equation (6-71)

$$\Phi_1 = 0.08166x^2 - 0.42028x + 0.01536$$

$$\Psi_1 = 0.14536x^2 - 0.16129x + 0.00695$$

Thus two possible roots are:-

$0.22702x^2 - 0.58157x + 0.02231 = 0$

i.e. $x = 2.52280, 0.03895$

or $0.06370x^2 + 0.25899x - 0.00841 = 0$

i.e. $x = -4.0980, 0.03224$

The only allowable values, then are:- $x = 0.03224$ and $x = 0.03895$

Clearly, the values which give the maximum output in the two allowable zones (in which cycling can take place) are the upper limits, and the values of x_1 to produce the minimum values of flow are the lower limits of the zones, namely 0.03224 and 0.137, 0.0199 and 0.03895, respectively.

These values of x_1 will not allow cycling and demand $\frac{dx}{dt} = 0$ for all t, so that in the practical case these values are not permissible.

(iv) Equation (6-93) could be solved for h in terms of x and t but the resulting expression is so complex as to be worthless.

No other possible roots to the equation are evident, so that there is no value of x in pond one, between the limits set, which will give the maximum output.

Consequently, pond one can be run at any value of x_1, the nearer it cycles to 0.137 however, the greater the output of brine from the system. The choice of x, would have to be made by economically balancing the output for various values of x_1 at $t = 0$, against the time it would take to get that value of x_1 in pond one, starting from sea water some period before the season.

In practice, the system is such that at $t = 0$, $x = 0.0199$ (sea water) throughout the whole system, and the condition of maximum output seems to be more related to the optimum depth calculations than these flow considerations. Consequently, the output has been calculated for this two pond case from $x = 0.0199$ at $t = 0$ for the depths of 12", 18" and 36" up to the time when pure oscillation takes place, and for the same systems after a long time to obtain the ultimate output to be expected if average weather conditions prevail continuously.

Now from equation (6-24)

$$F_2 = \frac{A_2 E_2 x_1 (1-k_1 x_2)}{(x_2 - x_1)}$$

Where $E_2 = (ax_2 + b) \sin t + (px_2 + q)$

and $x_2 = 0.137$ (25° Bé brine)

From § 3, inserting the values of a, b, p, q and given that on the average $k = 1.27$.

$$\frac{F_2}{A_2} = \left[0.052 + 0.171 \sin t \right] \frac{x_1}{(0.137 - x_1)} \quad ft^3/ft^2/month$$

If, however, $A_2 = A_1 = \frac{A}{2}$ where A is the total area in square feet, the output of saturated brine per unit of the total area A

$$= \frac{F_2}{A} = \left[0.026 + 0.0852 \sin t \right] \frac{x}{(0.137 - x)} \qquad \text{--- (6-94)}$$

The input of sea water, from equation (6-23) is:-

$$\frac{F_o}{A} = 0.0517 + 0.169 \sin t \quad \frac{(1-1.27x)}{(0.137 - x)} + (1.51 - 2.0x) \sin t$$

$$+ (0.865 - 3.56x) \qquad \text{--- (6-95)}$$

(cubic inches)/month.

For the quantities per unit of total area between times t_1 & t_2,

$$Q = \int_{t_1}^{t_2} \frac{F}{A} \, dt$$

∴ The output of saturated brine between the times t_1 and t_2 from the two pond system $= Q_2^*$

$$= \int_{t_1}^{t_2} (0.026 + 0.0852 \sin t) \frac{x}{(0.137-x)} \, dt \qquad \text{--- (6-96)}$$

but since x is a function of t given only by plots, equation (6-96) can only be evaluated graphically.

Similarly, the input of sea water between times t_1 & t_2 is the area under the curve of $\frac{F_o}{A}$ versus t within the limits of t_1 and t_2.

Using these expressions and the values of x from tables (6.7) and (6.8), tables (6.9) and (6.10) were constructed. These values are per unit of total pond surface area.

It can be seen from table (6.9) that the output of brine from the system practically does not vary with the depth of brine. This is what would be expected since evaporation rate is considered as a function of time and concentration only and to be independent of depth. In actual fact, evaporation will be a function of depth of brine as well (see § 3) with an optimum depth at each concentration,

so that in any rigid yield calculations this depth dependance of evaporation should be included.

The flow of saturated brine out is plotted against time in figure (6.13) for the various depths after considerable time has elapsed.

t	h = 12"		h = 18"		h = 36"	
	$F_{n/A}$	Q_2^*	$F_{n/A}$	Q_2^*	$F_{n/A}$	Q_2^*
0	0.0142	0.0000	0.0136	0.0000	0.0136	0.0000
$\frac{\pi}{6}$	0.0366	0.0267	0.0354	0.0265	0.0341	0.0262
$\frac{\pi}{3}$	0.0508	0.0705	0.0500	0.1701	0.0490	0.0670
$\frac{\pi}{2}$	0.0538	0.1213	0.0539	0.1209	0.0539	0.1209
$\frac{2\pi}{3}$	0.0465	0.1746	0.0471	0.1740	0.0476	0.1738
$\frac{5\pi}{6}$	0.0312	0.2177	0.0318	0.2170	0.0324	0.2168
π	0.0118	0.2353	0.0120	0.2343	0.0123	0.2340
$\frac{7\pi}{6}$	-0.0076	0.2370	-0.0077	0.2364	-0.0078	0.2351
$\frac{4\pi}{3}$	-0.0226	0.2219	-0.0227	0.2214	-0.0228	0.2210
$\frac{3\pi}{2}$	-0.0293	0.1944	-0.0290	0.1940	-0.0287	0.1939
$\frac{5\pi}{3}$	-0.024	0.1677	-0.0242	0.1673	-0.0236	0.1669
$\frac{11\pi}{6}$	-0.0089	0.1494	-0.0086	0.1488	-0.0082	0.1486
2π	+0.014	0.1515	+0.0136	0.1509	+0.0130	0.1505

Table (6.9) - Values of $\frac{F_n}{A}$ and Q_2^* for the Two Pond system for different depths of brine after a long period of time.

First year			Second year			Third year			Fourth year			Fifth year			Sixth year		
12"	18"	36"	12"	18"	36"	12"	18"	36"	12"	18"	36"	12"	18"	36"	12"	18"	36"
0.0044	0.0044	0.0044	0.0084	0.0071	0.0058	0.0111	0.0092	0.0070	0.0126	0.0107	0.0081	0.0134	0.0117	0.0090	0.0138	0.0124	0.0098
0.0272	0.0248	0.0219	0.0393	0.0340	0.0272	0.0463	0.0408	0.0321	0.0501	0.0454	0.0361	0.0522	0.0486	0.0387	0.0525	0.0504	0.0423
0.0079	0.0069	0.0059	0.0090	0.0086	0.0069	0.0108	0.0099	0.0079	0.0113	0.0107	0.0088	0.0116	0.0112	0.0094	0.0117	0.0115	0.0100
-0.0185	-0.0160	-0.0133	-0.0235	-0.0202	-0.0158	-0.0263	-0.0233	-0.0227	-0.0279	-0.0254	-0.0201	-0.0287	-0.0267	-0.0218	-0.0290	-0.0276	-0.0232
0.0084	0.0071	0.0058	0.0111	0.0092	0.0070	0.0126	0.0107	0.0081	0.0134	0.0117	0.0090	0.0138	0.0124	0.0098	0.0139	0.0128	0.0105
Seventh year			Eighth year			Ninth year			Tenth year			Eleventh year			Twelfth year		
0.0139	0.0128	0.0105	0.0141	0.0131	0.0109	0.0141	0.0133	0.0114		0.0134	0.0117		0.0134	0.0117			0.0119
0.0530	0.0516	0.0446	0.0534	0.0523	0.0464	0.0534	0.0529	0.0479		0.0534	0.0499		0.0534	0.0511			0.0520
0.0117	0.0117	0.0105	0.0118	0.0118	0.0108	0.0118	0.0119	0.0112		0.0199	0.0115		0.0120	0.0117			0.0119
-0.0291	-0.0282	-0.0243	-0.0292	-0.0287	-0.0253	-0.0292	-0.0289	-0.0261		-0.0287	-0.0258		-0.0288	-0.0263			-0.0266
0.0141	0.0131	0.0109	0.0141	0.0133	0.0114	0.0141	0.0134	0.0117		0.0134	0.0117		0.0134	0.0119			0.0121
Thirteenth year			Fourteenth year			Fifteenth year			Sixteenth year			Seventeenth year			Eighteenth year		
		0.0121			0.0122			0.0123			0.0124			0.0125			0.0126
		0.0529			0.0534			0.0539			0.0543			0.0546			0.0548
		0.0120			0.0122			0.0122			0.0123			0.0124			0.0127
		-0.0268			-0.0272			-0.0273			-0.0274			-0.0276			-0.0287
		0.0122			0.0123			0.0124			0.0125			0.0126			0.0129

Ditto

Table (6.10) Values of flow out of saturated brine from a system of two ponds from time t = 0 for the first eighteen years, with average weather conditions applying continuously. The second pond contains saturated brine at all times with the concentration of brine in pond one increasing from sea water at time t = 0 according to figure (6.12). On reaching the purely oscillatory part of the (x_1, t) curve, the output is the same for the previous year, and is marked "Ditto".

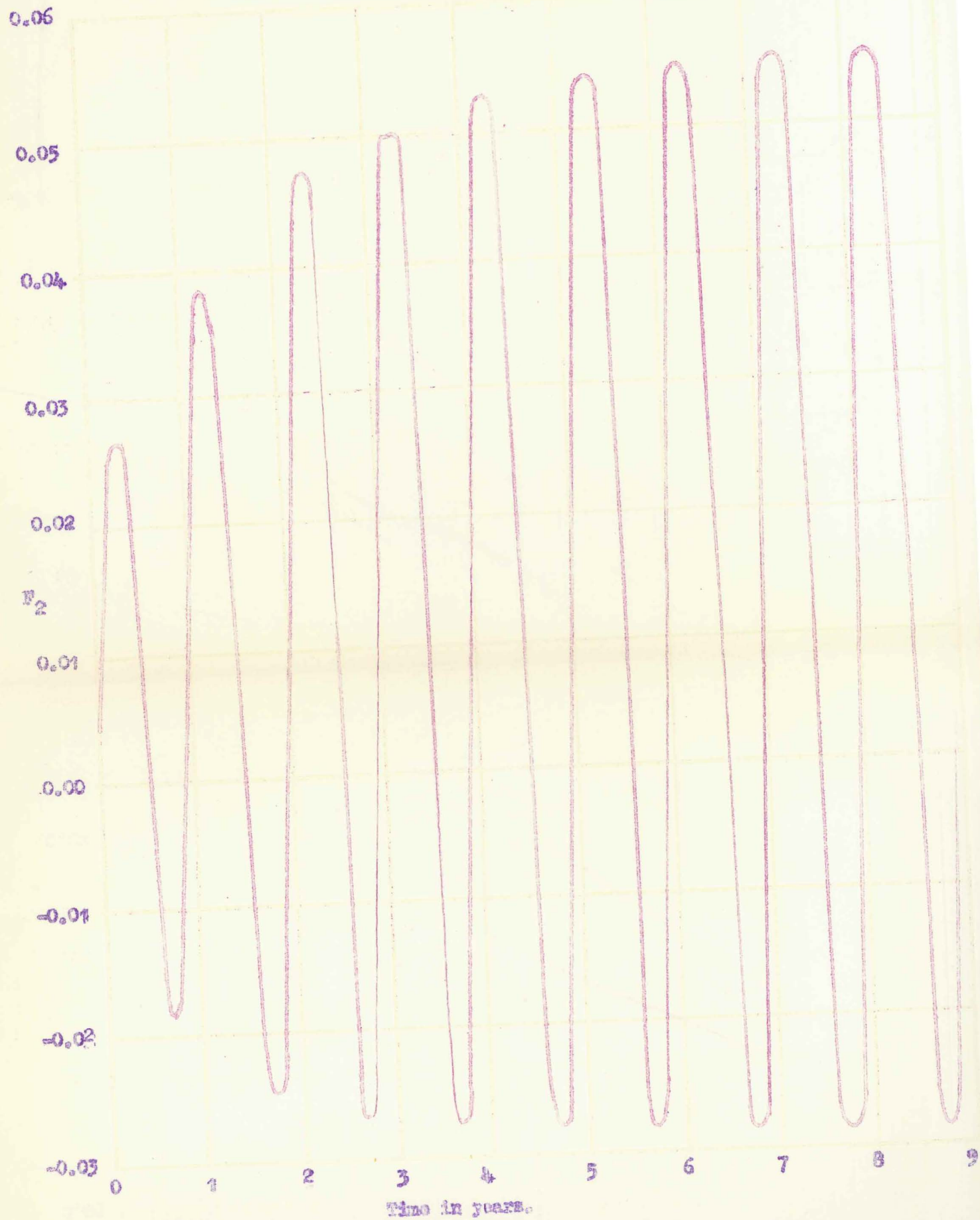


Fig. (6.12a) Flow from a two pond system for the first nine years when h = 12 inches

From the (x, t) curve of figure (6.12) it can be seen that the purely oscillatory curves, starting from sea water in pond one, and with saturated brine in pond two at all times, are reached in

9 years for the 12" depth system,

12 years for the 18" depth system, and

18 years for the 36" depth system.

Values of x were extracted from figure (6.12) and used to calculate the flow of saturated brine out from the two pond system for the first 18 years. Taking the 12" depth, the flows were plotted on figure (6.12a) to indicate the increase with time. The area under this curve was measured as the output of the system and table (6.11) gives the outputs of saturated brine from September to March in the first nine seasons of operation. These were plotted in figure (6.12b)

Season	Output from September to March.
1	0.0907 ft^3/ft^2
2	0.1287
3	0.1434
4	0.1650
5	0.1715
6	0.1732
7	0.1740
8	0.1748
9	0.1749

Table (6.11) - Outputs of saturated brine for 12" depth from September to March for the first nine seasons.

These results clearly show the build up of output for the two pond system over the early period of operation. This result is of great significance in that in establishing a salt works, a considerable number of seasons will be required before the system produces its ultimate yield of salt. This is of greater importance the less the evaporation, so that in the case of Grassmere, the Company would have to expect very poor yields for some years.

The above calculations were made for average weather

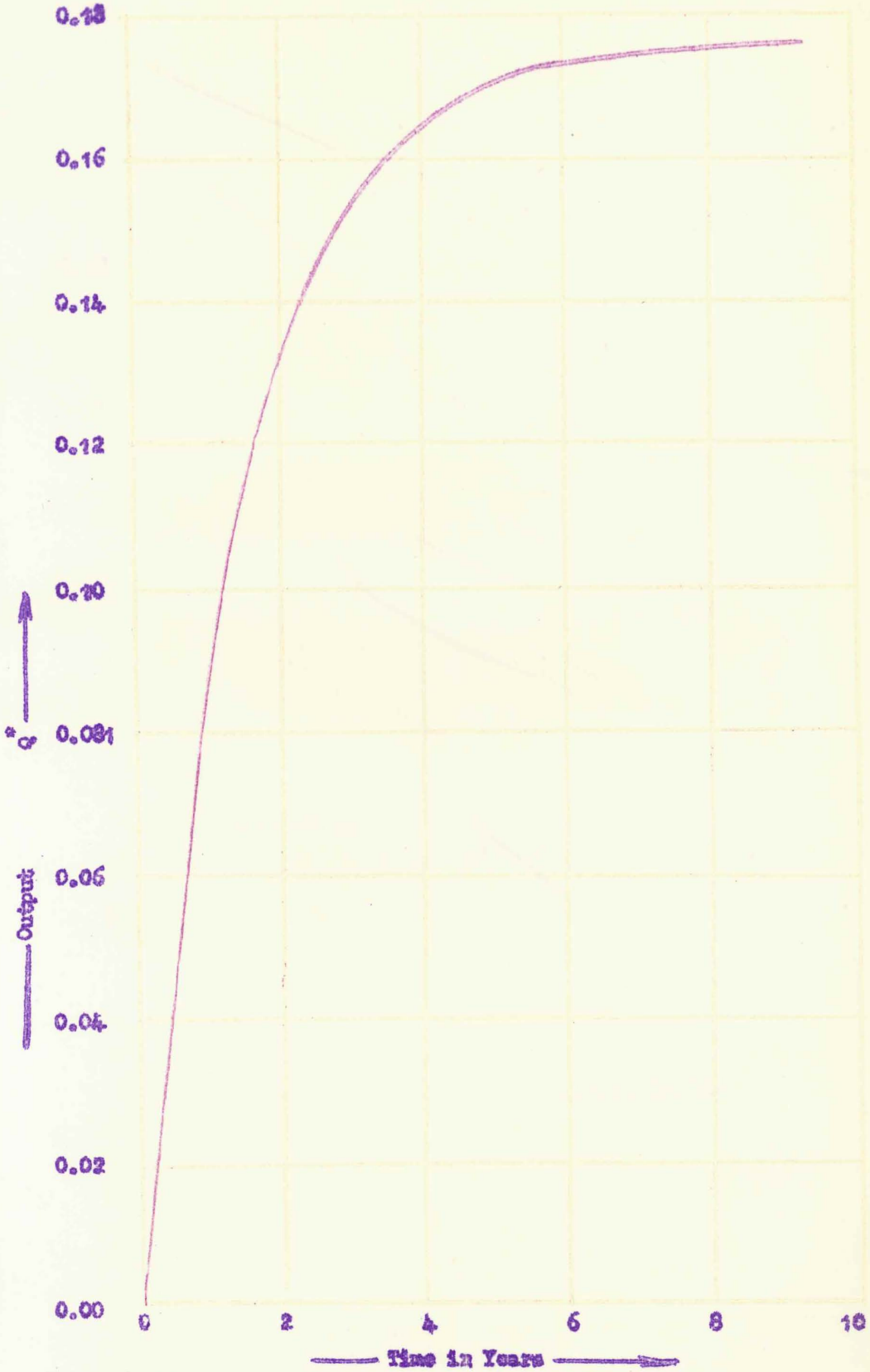


Fig. (6.12b) Output of saturated brine for the first nine years of operation for a two pond system.

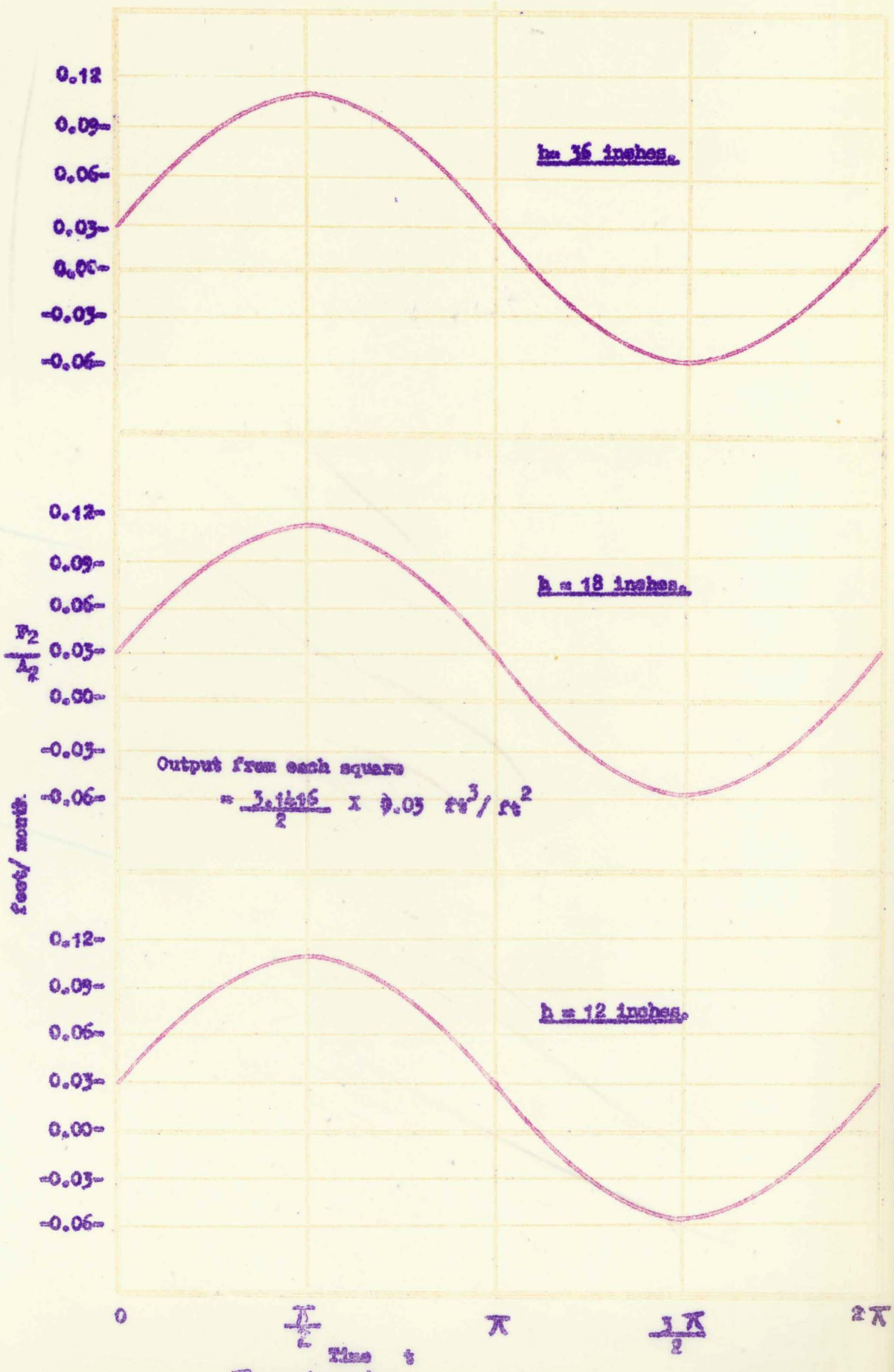

0.12
0.09
0.06
0.03
0.00
-0.03
-0.06
h = 36 inches.
h = 18 inches.
h = 12 inches.
$\frac{F_2}{A_2}$
feet/ month.
Output from each square
$= \frac{3.1416}{2} \times 0.05 \text{ ft}^3/\text{ft}^2$
0
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
Time t

Fig (6.13)

conditions applying continuously so that the situation would be more serious if weather conditions were worse than average for any season.

SOLUTION FOR MORE THAN TWO TANKS.

It is clear from the solution of the two pond case, that similtaneous solutions for x_1, x_2 ---------x_{n-1} as functions of t for the n pond case, is out of the question. A solution for 3 ponds could possibly be obtained by means of the differential analyser, if values of " Φ" and " Ψ" were calculated for values of x_1 and x_2, this requiring a great amount of numerical work.

Dr. L. Woods of Sydney University, suggested a relaxation method, applicable to the general case.

For pond i, generally -

$$\frac{dx_i}{dt} = f_i\ (x_o, x_1 \text{ -------}, x_i, x_{i+1})$$

and if x_i^j is defined as the value of x_i at time t_j, and f_{ij} as the value of f_i at time t_j, then -

$$f_{ij} = f_i(x_1^j, x_2^j \text{ -------- } x_i^j, x_{i+1}^j)$$

now: $$x_i^{j+1} = x_i^j + \left(\frac{\partial x_i}{\partial t}\right)^j n + \left(\frac{\partial^2 x_i}{\partial t^2}\right)^j \frac{n^2}{2} + \text{ ----------}$$

where n = the time interval between t_j & t_{j+1}

Similarly $$x_i^{j-1} = x_i^j - n \left(\frac{\partial x_i}{\partial t}\right)^j + \text{ -------}$$

$$\therefore\ x_i^{j+1} - x_i^{j-1} = 2n \left(\frac{\partial x_i}{\partial t}\right)^j + n^2 \left(\frac{\partial^2 x_i}{\partial t^2}\right)^j + \text{ ------ } 0(n^3) \text{ ------}$$

so that if n is taken small,

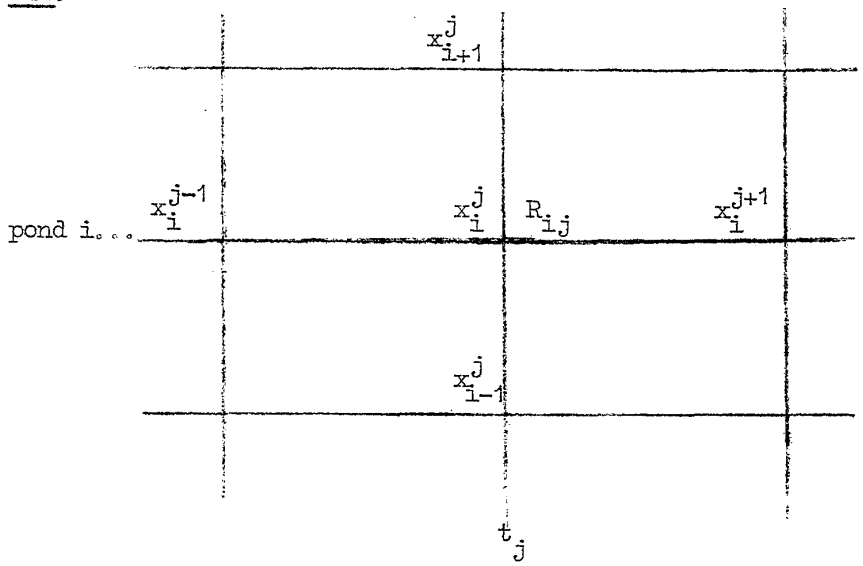
$$x_i^{j-1} - x_i^{j-1} = 2nf_{ij}$$

$$\therefore\ f_{ij} = \left(\frac{\partial x_i}{\partial t}\right)^j$$

Thus, a network could be set up from which the values of x in all ponds, and at all times, could be calculated by calculating the residuals R_{ij} at the point (i, j) from -

$$R_{ij} = x_i^{j+1} - x^{j-1} - 2nf_{ij}$$

Viz:-



$R_{ij} = 0$ for all i, j when the network is completely relaxed.

This net is bounded on three edges, $x_n = 0.137$, $x_o = 0.0199$ and $t_o = 0$ and at all these boundaries, the residual can be absorbed.

Thus, for five ponds, a network like that shown qualitatively in figure (6.14) would be obtained

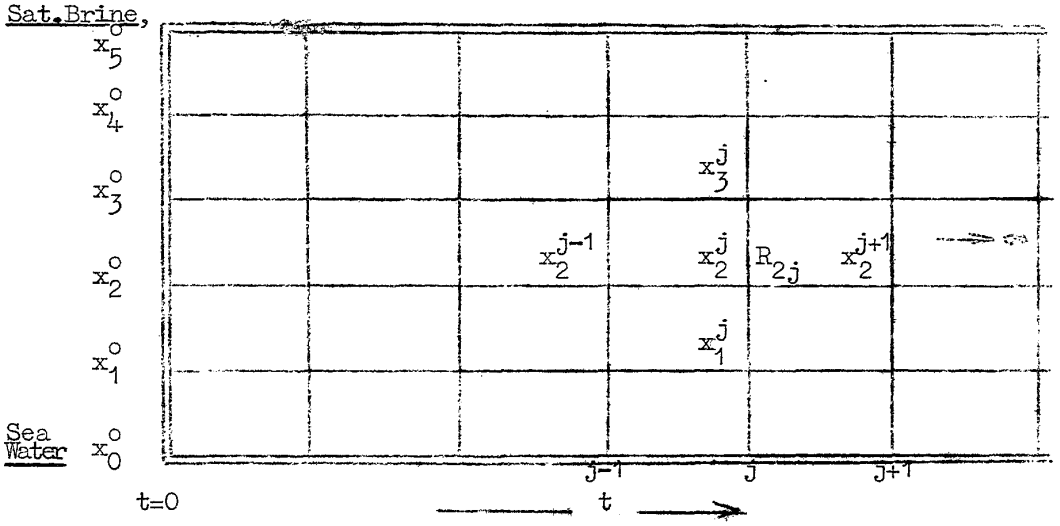


Fig (6.14) Relaxation network for 5 pond case.

Any arbitrary values of x could be used at time t = 0; after complete relaxation these would be adjusted to the correct values.

For the practical case, at t = 0 the concentration in all ponds is that of sea water, so that x_o, x_1, x_2 ------- x_5 could all be

set at 0.0199 and the network calculated until the purely harmonic function of x and t was obtained. This would then give the time to establish the ultimate output, the value of that output and the annual yield in the establishment of the oscillatory curve.

To utilise such a method, values of " Φ_r " and " Ψ_r " would have to be calculated for all values of x_o, x_1, x_2 ---------, x_{r-1}, x_r and x_{r+1} and if each of these could vary from 0.02 to 0.147 with increments of 0.001, and if t varied from 0 to 2π in increments of 0.01π, the number of complete calculations, for the 3 pond case (say), to establish a table for ready calculation of f_{ij}, would be of the order

$$(117)^3 \times 200 \sim 2.5 \times 10^8$$

and this is clearly impractical.

It seems that for more than a two pond system, solution of this problem is out of the question.

SOLUTION FOR THE INFINITE CASE.

From equation (6-47) -

$$l = \frac{1}{r}\left[\ln \frac{(px+q)^{p/q^2}}{x} - \frac{1}{qx}\right] + S$$

and the conditions,

$$x = 0.0199 \quad \text{when} \quad l = 0$$

$$x = 0.137 \quad \text{when} \quad l = L$$

will give the values of r and S

Also $p = -7.12, \qquad q = +1.73 \qquad (\S\ 3)$

$$\therefore\ l = \frac{1}{r}\left[-2.379 \ln \left(\frac{1.73-7.12x}{x}\right) - \frac{1}{1.73x}\right] + S$$

$$\text{i.e. } 0 = \frac{1}{r}\left[-2.379 \ln 79.815 - 29.044\right] + S$$

$$= -39.463 \frac{1}{r} + S$$

$$\text{i.e. } rS = +39.463$$

Also, from the second condition -

$$L = \frac{1}{r}\left[-2.379 \ln \left(\frac{1.73 - 7.12 \times 0.137}{0.137}\right) - \frac{1}{1.73 \times 0.137}\right] + S$$

$$= -8.278 \cdot \frac{1}{r} + S$$

Thus -

$$L = S\left(1 - \frac{8.278}{39.463}\right) \quad \text{or} \quad S = 1.266L$$

and $$r = \frac{39.463}{1.266L} = \frac{31.171}{L}$$

Hence -

$$^{1}/_{L} = 1.266 - 0.0763 \ln \frac{(1.73 - 7.12x)}{x} - \frac{0.0185}{x} \quad \text{--- (6-97)}$$

From equations (6-97) for $^{1}/_{L}$, and using the values,

$$a = -4.0$$
$$b = +3.02 \quad \text{------ (§ 3)}$$
$$p = -7.12$$
$$q = +1.73$$

table (6.12) was constructed for evaluation of equation (6-57)

x	inches/month $1.73 - 7.12x$ $px + q$	inches/month $3.02 - 4.0\,x$ $ax + b$	$^{1}/_{L}$
0.0199	1.5883	2.9404	0.0000
0.025	1.5520	2.9200	0.2110
0.030	1.5164	2.9000	0.3500
0.040	1.4452	2.8600	0.5298
0.050	1.3740	2.8200	0.6431
0.060	1.3028	2.7800	0.7228
0.070	1.2316	2.7400	0.7829
0.080	1.1604	2.7000	0.8306
0.090	1.0892	2.6600	0.8702
0.100	1.0180	2.6200	0.9040
0.110	0.9468	2.5800	0.9336
0.120	0.8756	2.5400	0.9602
0.137	0.7546	2.4720	1.0000

Table (6.12) - For Evaluation of equation (6-57)

The values of $(px + q)$ and $(ax + b)$ were plotted against $^{1}/_{L}$ in figure (6.14a) for graphical integration of equation (6-57)

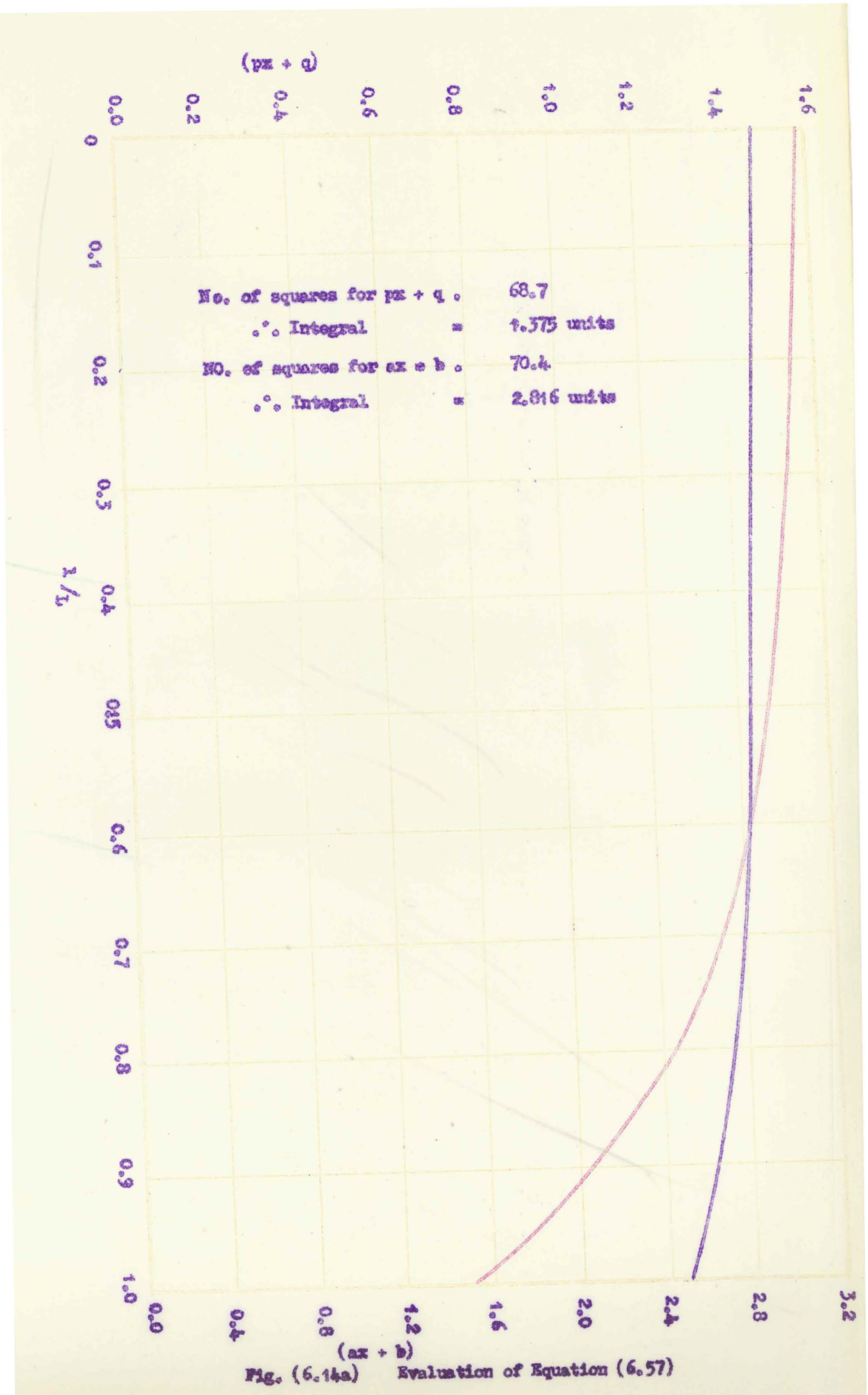


Fig. (6.14a) Evaluation of Equation (6.57)

Since from (Ref. 29)

$$\rho_n = 1.2110 \text{ for } 25^\circ \text{ Bé brine}$$

$$\rho_o = 1.0258 \text{ for sea water}$$

$$\rho_w = 1.0 \text{ for pure water}$$

$$\frac{\rho_n - \rho_o}{\rho_o - \rho_w} = 7.16$$

and hence from figure (6.14) in equation (6-57)

$$\frac{F_n}{A} = 0.0331 \sin t + 0.0161 \quad ft^3/ft^2/\text{month} \quad \text{---- (6-98)}$$

and $$F_o = 8.120\, F_n$$

$$\therefore \quad \frac{F_o}{A} = 0.269 \sin t + 0.1307 \quad ft^3/ft^2/\text{month} \quad \text{---- (6-99)}$$

Where A is the total area in square feet.

For the quantities, per unit area, between times -

$$t_1,\ t_2,\ Q = \int_{t_1}^{t_2} \frac{F}{A}\, dt$$

$\therefore$ The output of saturated brine from an infinite series per unit area from time $t = 0$ to time t

$= Q^*_\infty$, so that if $Q^*_\infty = 0$ for $t = 0$

$$Q^*_\infty = \left[0.0331\ (1 - \cos t) + 0.0161\ t\right] \frac{6}{\pi} \quad \text{---- (6-100)}$$

If Q_∞ is the input of sea water into the same system between the same time limits:

$$Q_\infty = \left[0.269\ (1 - \cos t) + 0.1307\ t\right] \frac{6}{\pi} \quad \text{---- (6-101)}$$

where units are ft^3/ft^2 total area.

The flows and outputs of saturated brines from this infinite system calculated from equations (6-98) and (6-100) are given in table (6.13). The equivalent values of $\frac{F_o}{A}$ and Q^*_∞ are also given.

Table (6.13) - Flows and outputs for an infinite system under continual average weather conditions.
Assumes $\frac{dx}{dt} = 0$. Time basis 1 month.

DATE	t	$F_o/_A$	$F_n/_A$	Q^*
Mid Sep.	0	0.131	0.0161	0.0000
Oct.	$\frac{\pi}{6}$	0.265	0.0327	0.0246
Nov.	$\frac{\pi}{3}$	0.362	0.0448	0.0638
Dec.	$\frac{\pi}{2}$	0.398	0.0492	0.1115
Jan.	$\frac{2\pi}{3}$	0.362	0.0448	0.1593
Feb.	$\frac{5\pi}{6}$	0.265	0.0327	0.1984
Mar.	π	0.131	0.0161	0.2231
Apr.	$\frac{7\pi}{6}$	-0.0034	-0.0005	0.2307
May	$\frac{4\pi}{3}$	-0.101	-0.0126	0.2236
June	$\frac{3\pi}{2}$	-0.136	-0.0170	0.2082
July	$\frac{5\pi}{3}$	-0.101	-0.0126	0.1925
Aug.	$\frac{11\pi}{6}$	-0.0034	-0.0005	0.1856
Sep.	2π	+0.131	+0.0161	0.1933

All values of Q are cumulative.

Collecting the output information together by plotting the output of saturated brine per unit of total area produces figure (6.15).

It is clear from this figure that a two pond system gives a considerably greater yield than does a one pond system. In the six months September ($t = 0$) to March ($t = \pi$) the outputs are 0.0858 and 0.1232 ft^3/ft^2 respectively, a difference of 43.6% on the one pond value.

However, it is also clear that the calculations for the infinite system are erroneous since although over the whole year the

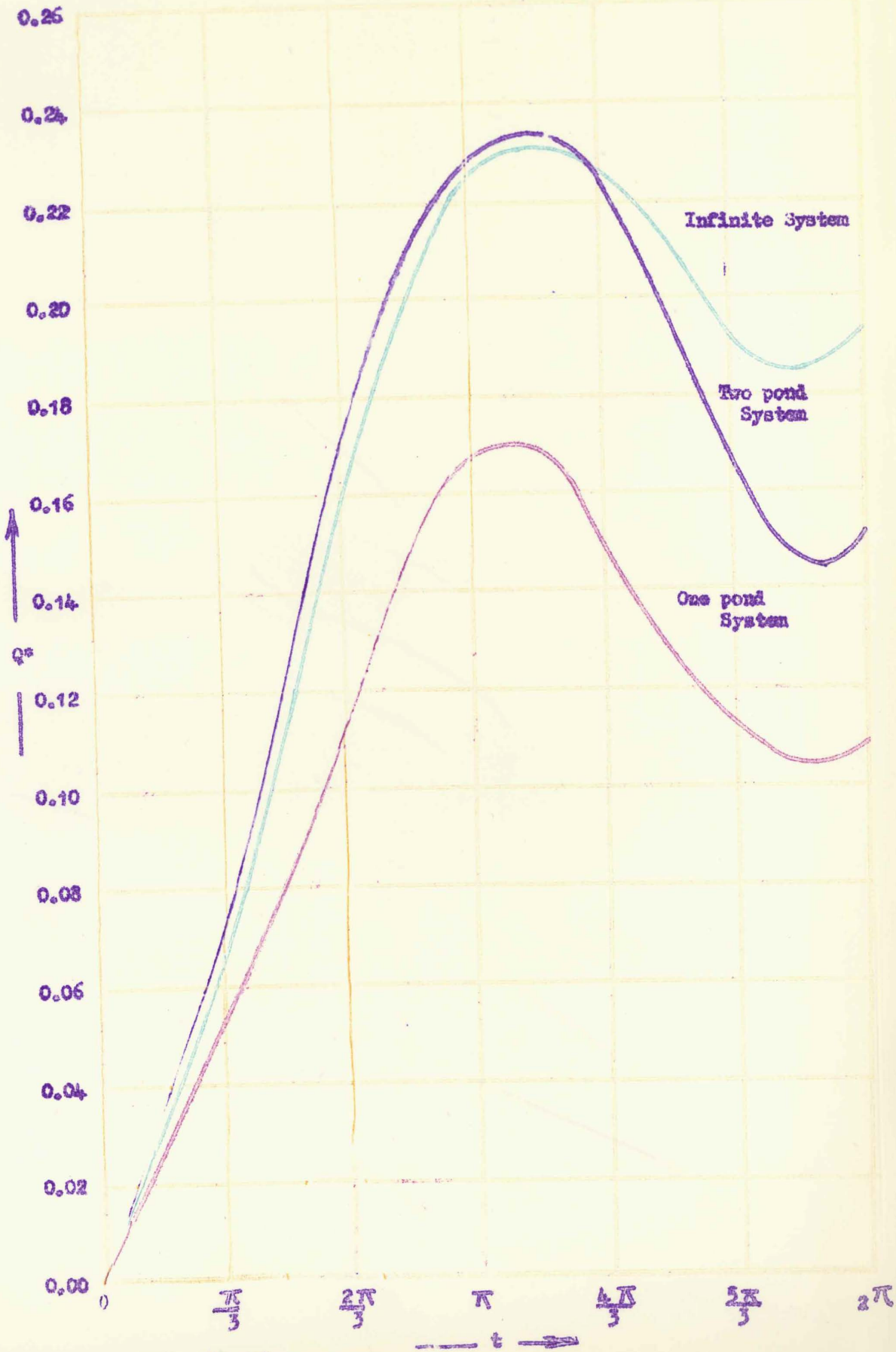


Fig. (6.15) Plot of output versus time for a one pond, two pond and infinite pond systems.

output from the infinite case is greater (namely 0.11 as compared to 0.079 for the 2 pond system), from September to March the output of the infinite system lags the other by up to 5½%. This has resulted from what now appears the erroneous assumption of $\frac{dx}{dt} = 0$ for the infinite case. $\frac{dx}{dt}$ is undoubtedly small here but is large enough to make a significant contribution to Q^*_∞. It may well be that the difference in output between two pond and infinite pond systems is small, but a rigid derivation of $\frac{dx}{dt}$ as a function of x, h and t for the infinite case will be necessary to evaluate the difference. Consequently, no valid conclusions can be made involving the infinite case as derived here.

From the results of the one pond and two pond calculations however, conclusions can be drawn since the derivations of the relevant equations was rigid if the assumptions of average weather conditions and independence of evaporation rate on brine depth are legitimate.

IV.

C O N C L U S I O N S.

The two important sections of work investigated, are on the rate of evaporation and the variation of output of saturated brine from a system of ponds under average weather conditions.

(i) The expression for evaporation rate is based on the assumption that natural evaporation from a brine solution is a function of the brine concentration and the weather conditions only. It also assumes that the weather is completely defined by the time of the year, so that average weather conditions prevail at all times and any unusual spells of weather are not allowed for. An investigation of the mechanism of evaporation and energy balances involved, show that evaporation is also a function of brine depth. The derived relation of E to x and t ignores this dependence which is clearly important, but has been used in the absence of any better expression. It is clearly important to establish actual values of evaporation rates from brines of different concentration and of different depths at all times, so that a more reliable expression for E can be derived. The uncertainty of the available data, and the complete absence of any information relating evaporation to brine depth, makes it impossible to derive a more reliable expression. The data must be available before any further advance in this field can be made.

The residence time calculation indicates that a unit of brine will require between one and two years to be saturated from sea water, when in a system of an infinite number of ponds. For a finite number, the time would be greater.

Applying this to Grassmere, it is apparent that every effort must be made to increase the evaporation rate, and since the unit of sea water must remain in the system for at least one Winter, every effort should be made to ensure that its dilution by rain water is a minimum.

Working at optimum brine depths and possible addition of dye to the brine will help the increase in E, while deep storage of concentrated brines, over the Winter months, will ensure minimum dilution.

(ii) The general problem of flow and output of saturated brine from a pond system, when average weather conditions prevail, has been solved rigorously for a one-pond and a two-pond system only. It has been shown that the output from two ponds is about 45% greater than that from one pond of the same total area over a season.

For the case of an infinite number of ponds, the calculations were based on an erroneous postulate of the rate of change of concentration at any point with time being zero. In fact this $\frac{dx}{dt}$ is small and making it zero may only involve a small error.

The evidence is not conclusive, but from the calculations presented, it would appear that sub-division of a given total pond area into a large number of ponds, involving a large capital expenditure, is unwarranted and that possibly very few more than 2 ponds would produce as great an output as an infinite number. The number of ponds also affects the number of seasons required for a system to reach the stage of producing its regular yield. The time decreasing as the number of ponds increases. Thus in designing a salt works, an economic balance would be required, based on the output over a given total number of years, to decide the best number of ponds.

The structure and formation of the land on which the works was to be constructed would, of course, be a practical factor in determining the number of ponds.

It would be incorrect to conclude that the output from a given system is independent of the brine depth, h, since in this work the evaporation rate has been considered independent of h.

The suggested energy balances, when solved to give an optimum depth and the relation between E and h, will provide information on the optimum output. The attempt made in this investigation to maximise the flow, could not produce a solution since there is only one set of conditions which can ultimately apply for x in pond 1.

It should be noted that from this work, the time to reach best

output from a system has been shown to depend on the depth of brine, so that the optimum depth for best output (i.e. best evaporation) will be decided on an economic balance again.

The solution of the equations for the two pond case presented many problems. The (x, t) curves resulting from numerical solution on the differential analyser, show that a strictly periodic function of x and t results eventually, independent of the depth of brine. The establishment of this curve is important since it shows that the pond system will have only one set of operating conditions when at equilibrium. This means that the ultimate yield of salt from an area, providing average weather conditions prevail continuously, is fixed by such design parameters as number, size and shape of ponds and depth of brine. Optimum values must be found for an economic design.

The solution of the equations arising for more than two ponds by the procedure followed for the two pond case is clearly impractical. However, if the limits of concentration for any pond can be calculated in the manner given, the final periodic curve may be able to be defined fairly closely and approximated to as in the appendix. The ultimate output could then be estimated. Alternatively from the form of the xvt curves after a long period, it is clear that at considerable brine depths (of the order of 18 inches and greater) the curve could be represented very closely by $x = \chi(h) \sin t$, where $\chi(h)$ is a function of h only. It may then be possible to solve the infinite (and general) case rigorously for deep brine in ponds.

Another important conclusion which can be drawn from the (x, t) curves is that due to the long period of time required to establish the ultimate periodic curves, particularly at considerable brine depths, it may be many years before a large output of salt can be expected, and that the output should increase with time up to that point. This is only true if the weather conditions remain "mathematical" i.e. average, for, due to sudden abnormalities in the weather, the output might be suddenly increased or decreased.

It is important to note that if at the beginning of each season, the system is started up with saturated brine in pond 2 and sea water in pond one, the output will never rise above that given in the period 0 to 2π .

It should be noted that the calculations made were based on the assumption that the concentration of brine in the final pond was saturated. That is at the beginning of a season (i.e. when $t = 0$), saturated brine would be available for flow out of the system immediately. Two points arise from this:

(a) In practice, at the starting up of the system, all ponds would contain sea water and no flow out of saturated brine would be obtained. The set up then is different from the conditions applied above, and in particular, the time to reach the ultimate output will be different, since there is now no flow through the system and only flow in. The times given to reach ultimate output for the two pond system apply only when saturated brine is in pond number two at $t = 0$ and at all t. This is possible but is not as found in practice.

(b) From the curve of output v. time (Fig. 6.15) for the one pond and two pond systems, it is seen that production of saturated brine from September to about the beginning of April, would be 0.0875 and 0.1247 ft^3 of saturated brine/ft^2 of total area, respectively. After September, saturated brine would have to be pumped back into pond 2 to maintain its concentration at saturation, which is of course impractical, or the concentration of pond two would drop below saturation and would not be saturated again in the following September. This output curve would not apply then to the following season.

Alternatively, removal of saturated brine from pond two could stop about early January when the outputs would be 0.056 and 0.079 ft^3/ft^2 for the one pond and two pond systems respectively. Further evaporation would then cause crystallisation of salt in the final pond rather than production of saturated brine, until April, when dilution by Winter rains would bring the concentration back to saturation in September.

Clearly these alternatives are the maximum and minimum limits of output, for, with deep storage in the Winter resulting in less dilution of the brines, some intermediate value of output could be selected greater than the minimum but less than the maximum, so that saturated brine would be available at the beginning of the following season.

For the two pond system, stopping production of saturated brine in mid February would mean a yield of 0.1217 ft^3/ft^2.

In practice, this is when production does stop. The remaining brine could be further concentrated by continuing the process, but with no output, or be placed in deep storage with other brines depending on what is found to be best. At all costs, some saturated brine must be available the following September for these curves to apply.

For Grassmere, the total concentrating area, including pickle ponds, is of the order of 1300 acres (56×10^6 ft^2) so that for a two pond system there, the output in an average year from September to mid February $\sim 6.82 \times 10^6$ ft^3

$\sim 4.72 \times 10^8$ lb of solution (Ref. 29)

(iii) The mechanism of evaporation and the energy balances related to it can provide much valuable information. Although this aspect has not been advanced very far in this investigation, it is clear that the results available by the solution of the energy losses etc., would be very valuable. Optimum depths for the different brine concentrations would clearly result in more economic working and far better control.

This optimum depth results from a balance of energy absorbed, which increases as the depth increases and pond temperature, which decreases as the pond depth increases. The losses can be related in some way to the depth of brine since they depend on the temperature, and these losses must be maintained at a minimum. It has been established in this work that the usual assumption of the pond bottom being a best insulator is valid since instantaneous energy losses are negligible. Over an infinite time there will be a nett gain of energy through the

pond bottom due to the general cooling of the earth, although this clearly is of no practical significance.

OTHER GENERAL CONCLUSIONS.

(i) From the summary of the literature on evaporation, as given in the appendix, it appears that the aerodynamic approach to calculating evaporation from large ponds by using relatively easily measured physical quantities is the most promising. The empirical approach is shown to be completely inadequate and the energy balance procedure will only be of value if some exact method of eliminating the surface temperature of the pond is developed. Combination of the energy balance and the empirical approaches gives no better result than the empirical relations alone. The fundamental concepts on which these relations are based are so in error that their use is hardly justifiable. Consequently, the exact relations involved in the aerodynamic approach suggest its close study for application to a system where there are a number of large ponds close together and containing sea water brines of different concentrations. If reliable factors relating the rate of evaporation from brines of different concentrations can be derived, there should be little difficulty in applying this method to the solar salt industry.

(ii) There appears little possibility of establishing accurate correlations between the evaporation from evaporimeters and large ponds, over a complete range of conditions. Consequently, evaporation measured in evaporimeters and applied to a large pond will, under many conditions, produce appreciable error. Nevertheless the evaporimeter remains, at the moment, the most satisfactory method for assessing evaporation rates.

V. SUGGESTIONS FOR FUTURE WORK

Based on the results of this work, a number of suggestions for future investigations can be made.

(i) It is clearly important that a reliable method of assessing evaporation rates be developed so that some accuracy can be

expected of calculated results. Consequently, of paramount importance is the result of investigations on the aerodynamic approach now being undertaken by Bonython and Thorthwaite and Holzman. It is suggested that the relation between the evaporation from pure water and from brines of different concentrations under identical conditions, be investigated thoroughly so that measured values of evaporation at one concentration can be used to calculate values at other concentrations. It is expected that such factors will be functions of brine concentration and of time (or weather). These factors should be determined for a particular site many years prior to the establishment of a solar salt works as all design calculations will be based on them.
Once such an evaporation relation has been established, the evaporimeter can be relegated to its proper place - as a tank for investigation work only.

(ii) It is recommended that the energy balances and mechanism of dissipation of energy be the subject of a very thorough investigation. If a valid method of eliminating the surface temperature of a brine pond can be found, such energy balances could be used as a precise method of calculating evaporation continuously from a body of brine. However, direct value can be predicted from such study in establishing optimum depth relations to produce the greatest evaporation under given meteorological conditions, and hence the best output of saturated brine. It is considered that this investigation also is of great importance to economic operation of the salt industry.

(iii) With more reliable data and the results of the investigation of the dependance of E on h, an accurate expression for E should be derived.

(iv) It is suggested that the case of an infinite number of ponds be rigorously investigated to establish just how necessary it is to subdivide the pond ares. It is certain that $\frac{dx}{dt}$ is small and it may be possible to simplify the investigation by expressing it in the form $\left(\frac{\partial x}{\partial t}\right)_1 = \chi(h) \sin t$ where $\chi(h)$

is some small valued constant depending on h. Such an expression could give either positive and negative values of $\left(\frac{\partial x}{\partial t}\right)_l$.

The evaluation of χ (h) appears difficult although $\frac{\partial x}{\partial t} = 0$ for a number of values of x, t and l which may be definable.

The solution for three ponds or more appears out of the question unless the ultimate oscillatory curve can be obtained. If this is possible, the solution of the three pond case would be valuable.

The solutions should also be repeated for different values of the ratios of pond areas to each other to see if any optimum ratio exists. This investigation has shown that the ratio should be less than 1 for the two pond case and the true value should be determined. Similarly, for other systems.

(v) An investigation of systems in which at time t = 0 contained only sea water would lead to the time to reach the ultimate output, more closely related to the actual case.

In such a system, the flow out of saturated brine is zero and the material balances resulting from putting $F_n = 0$, $\frac{dx_n}{dt} \neq 0$ when solved would give this result.

In general, the type of equation to be solved is of the form of equation (6.33) for $\frac{dx_r}{dt}$, itself a function of (x_{r-1}, x_r, x_{r+1} ------x_n) and t. Consequently solution is going to be most difficult.

All these calculations should be based on the new expression for E in terms of x, t and h.

(vi) An investigation on the crystalisation of NaCl from the saturated brine should be carried out. The limit to the concentration of brine before magnesium salts deposit is of the order of 29° Bé, and based on the flow of brine out from the concentrating area, yields of salt could be calculated. Principally, such investigation would resolve around evaporation from a brine pond of some concentration between 25 and 29° Bé with flow in only.

The method of harvesting would be a factor in the form of the crystallising area.

(vii) Other general points which should be looked into are:

(a) The effect of sudden weather changes on output and whether such change will prevent the system reaching its ultimate output.

(b) The effect of deep storage on output and the relation between the brine requirements of the next season, the amount of saturated brine removed this season and deep storage.

V1.

NOMENCLATURE.

A = Surface of a pond.

$\mathcal{A}$ = Austaush Coefficient of turbulent mixing.

a = A constant in evaporation formula.

b = A constant in evaporation formula.

C = Constant in Dalton's Equation.

c = A general constant

E = Evaporation rate of water from any body.

F = Evaporation factor $= \frac{\text{Gross evaporation of brine in a pond}}{\text{Gross evaporation of water in a tank}}$

F = Volume flow rate.

f = Function.

H = Absolute humidity of the air.

h = Depth of brine in a pond.

h_{opt} = Optimum depth of brine in a pond.

h' = Path length of energy waves in a pond.

h_c = Convection heat transfer coefficient.

I_λ = Intensity of incident radiation of wavelength λ.

i = A general constant.

K = Mass transfer coefficient.

k = The ratio: $\frac{\text{Total weight of all salts in the brine}}{\text{Weight of NaCl in the brine.}}$

k_s = Coefficient of heat transfer by conduction of the soil.

l = A general length parameter.

L = Length of a finite area considered to contain an infinite number of ponds.

L_w = Latent heat of evaporation of water.

m = Weight of water evaporated.

n = Pond number, 1 to n.

p = Vapour pressure of water vapour from a liquid or the partial pressure of water vapour in the air.

Q = Input of sea water into a pond system.

Q^* = Output of saturated brine from a pond system.

q = Moisture content of the atmosphere.

R = Bowen's Ratio. $\frac{\text{Sensible heat lost}}{\text{Latent heat lost}}$

r_λ = Reflectivity of a surface to energy waves of wavelength λ .

S_p = Specific heat of brine between the limits of temperature employed.

t = Time

t_r = Residence time.

u = Wind velocity.

V = Weight rate of flow of brine.

W = A weight term in general.

x = Weight fraction of concentration

$$= \frac{\text{Weight of NaCl in a sample of brine}}{\text{Total weight of the brine sample.}}$$

Z = Height in the atmosphere above the evaporating surface.

$\alpha_{p_{\theta_a}}$ = Absorbtivity of the pond (surface) to radiation of such wavelengths as is radiated from the atmosphere at temperature θ_a

α = Angle of incidence of an incoming energy wave.

γ = Ratio of areas of ponds in a system.

δ = A small increment in any variable.

ε_p ε_{at} = Emissivity of the pond surface and of the atmosphere respectively.

ε = $E_{nett} - E_{nett\ average}$

$\psi(x)$ $\Phi(x)$ = Functions of x in the expression $h.\frac{dx}{dt} = (x)\sin t + (x)$

λ = Wave length of radiant energy.

μ = Coefficient of absorption of radiant energy for Lambert's Law.

η = Refractive index of the media.

π = 3.141593

ρ = Density of the brine.

ρ_ω = Density of water = 62.4 lb/ft^3.

σ = Steffan's constant. (Ref. 57)

θ = Temperature ($^\circ$F generally)
$(\theta + 460)$ = $^\circ$F absolute.

SUBSCRIPTS.

0 = flow entering the first pond or system of ponds.

1,2,3,----r,----n = Flow leaving the first, second, third, --- -------- the rth, -------- the nth pond in a system of n ponds.

a = Air.

a^t = Atmosphere.

b = Brine.

g = Ground.

p = Pond

s = Brine or water surface.

w = Water.

V11.

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APPENDICES.

Appendix for § 3.

ESTIMATION OF EVAPORATION RATES FROM LARGE PONDS CONTAINING WATER.

Direct measurement.

(i) The method of direct measurement of evaporation from large expanses of water was early abandoned, due to the overwhelming practical difficulties. (Ref. 12 & 13). Stilling tanks have been used to overcome ripples and waves on the surface, but in areas where strong prevailing winds exist, a build-up of water will take place in lee of the wind. This "build up" will be uncertain, due to variation in wind velocity (e.g. gusts etc.) and will vary from one time period to another, so preventing any estimation of level change due to evaporation of water. Further, in the case of solar evaporation of brines for salt manufacture, brine is flowing through the system of ponds, and this flow may vary considerably in a time period, which will affect any determinations of evaporation. Also, this method is very dependent on any gains of ground water to, or losses of brine from the pond by seepage. If evaporation was determined by another procedure, some estimate of such seepage could be made, whereas by this direct measurement method, neither evaporation nor seepage can be estimated with any accuracy at all.

(ii) #### Measurement in evaporimeters.

It was very early in the development of evaporation measurement methods that the suggestion of measuring the evaporation in small tanks arose, as rainfall and incident solar energy over an area are measured. (Refs. 4, 5, 6, 12, 15) Clearly, in testing these small tanks, called evaporimeters, a comparison of the evaporation from one has to be made with the evaporation from a large pond to determine whether the evaporations are comparable. That is, it has to be established that the evaporation measured in an evaporimeter situated near a large expanse of

water was comparable with the evaporation from the large area itself. The difficulties enumerated in (1) above still apply, and consequently the correlation of the two values has been neglected by many workers.

In general, the evaporimeter is a small circular tank with depth approximately equal to the diameter, which may be above the ground or buried to various levels in the ground. Measurement of evaporation is made by either taking level readings at time intervals with some form of level gauge or by adding a weighed amount of water until the level returns to a fixed point.

The great diversity of designs which are clearly possible from this "definition" of an evaporimeter, has been a serious set back to its use and Foley (Ref. 66) has discussed the problems associated with evaporimeters.

Sutton (Refs. 5, 32), Sheppard (Ref. 6) and Rowher (Ref. 9) have pointed out the need for standardisation of evaporimeters in that it is established that "evaporation rate from an exposed surface varies with the dimensions of the surface". Bonython (Ref 4) and Sheppard (Ref. 6) have also stated the conditions for an ideal evaporimeter: For an evaporimeter to give a true reading of evaporation from a surrounding land or water surface it must:-

(a) Have a surface which is flush with the surrounding surface of land or water, so that there is a minimum interference with the normal horizontal wind movement over the surface;

(b) Have a roughness parameter identical with the surroundings;

(c) Have heat losses and gains per unit volume of liquid the same as heat losses from the large body. This means the heat balances of the two must be identical.

(d) Have the vapour pressure at the surface of the liquid maintained at the same value as the surroundings. Thus, the temperature and temperature gradients in the liquids must be maintained identical.

It is clear that such requirements will be most difficult to

meet, but standardisation will at least eliminate some of the diversity.

Details of an Australian standard evaporimeter have been given (Refs. 4, 31). This is a sheet metal cylindrical tank 3 feet in diameter and 3 feet deep, set inside a similar tank 4 feet in diameter sunk into the ground with its rim level with the surface. The rim of the inner tank is 2" above that of the outer one, the inner tank is filled with water to within 3 inches of the rim and the outer one to the top. (Fig. A3.1)

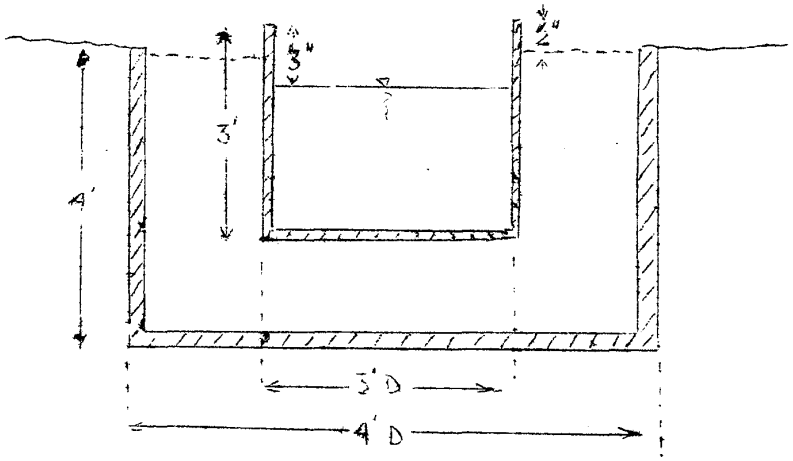


Figure A3.1 - Australian Standard Evaporimeter.

The outer "guard ring" assists in producing uniform conditions in the inner tank, in which evaporation is measured, and generally a minimum interference of wind structure is found with these dimensions.

Bonython, (Ref. 4) has carried out a series of experiments indicating the effect on evaporation rates from a standard evaporimeter of such factors as:-

(a) Surface treatment of the metal of the evaporimeter.

(b) Working level of the water surface.

(c) Presence of wire netting bird screens over the surface.

(d) Impurities, such as dust or oil on the surface.

Having thus defined the evaporimeter, it is necessary to investigate the correlation of evaporation from it and the surrounding areas.

It is clear from § 1 and § 2 that the exposure of the two

bodies of water must be identical. The meteorological conditions must be identical above them both. This means that the situation of the evaporimeter relative to the pond (i.e. the large expanse of water) is most important. Figure (A3.2) shows that in (a) air approaching saturation flows over the evaporimeter whereas in (b) it does not, so that here the wind direction could affect the relative evaporations. (c) overcomes this source of error but here the evaporimeter must be constructed at a higher level than the surface of the pond and will produce a wind structure above the evaporimeter, different to that above the pond surface. A perfect combination can not be obtained.

From the energy point of view the walls of the evaporimeter seriously affect the heat balance. Heat which would not be absorbed in the same volume of liquid in the pond, is absorbed through solar energy striking the walls and being reflected inside the evaporimeter.

Block (et al) attempted to overcome some of these difficulties by lining evaporimeters with mirrors. (Ref. 33)

It seems then that the evaporation measured in any one tank will be the same as the evaporation in any other, only if the tanks are identical, they are identically situated and then only when identical conditions apply above them both. Bonython (Ref. 23) carried out an extensive investigation on two evaporimeters, one an Australian standard and another 10 ft in diameter by 3 feet deep, both situated at the same site and another investigation of three evaporimeters at different sites: (Australian standard evaporimeters at The Waiti Institute, Dry Creek and Adelaide). His results clearly show the difference in measured evaporation from two different sized ponds identically situated (summarised in Table A3-1) and the difference in evaporation from fairly similar evaporimeters at 3 different sites separated by less than ten miles. (Typical example in Table A3-2)

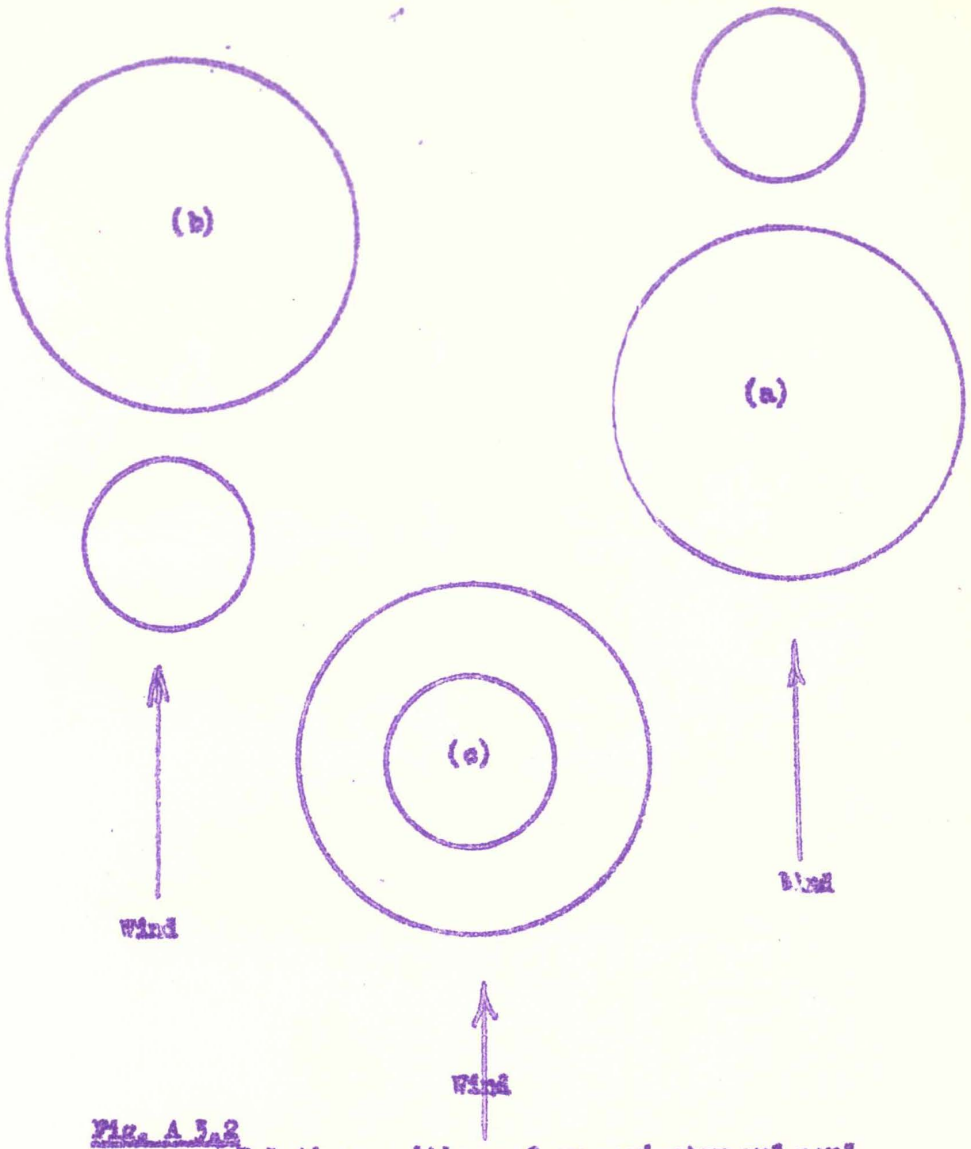


Fig. A 3.2 Relative positions of evaporimeter and pond.

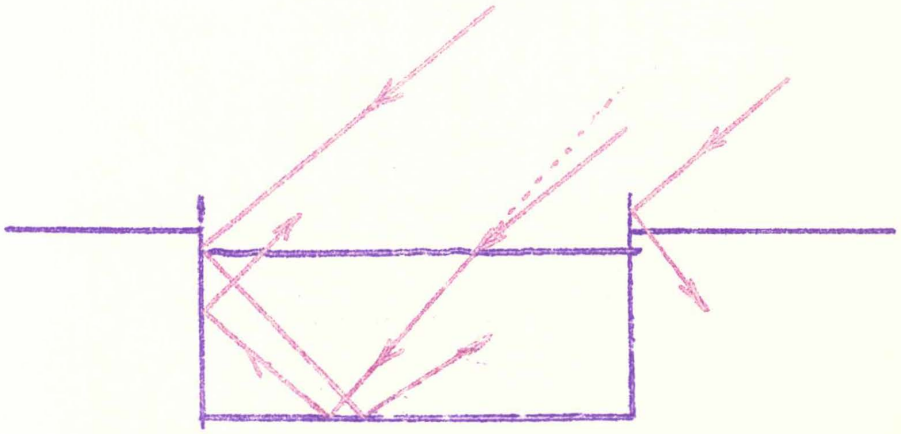


Fig. A3.3. Energy mechanism in an evaporimeter.

Table A3 - 1 - Difference in evaporation from 2 tanks situated at Dry Creek. (Ref. 23)

DATE 1949	Evapn. from Standard cm/28 days	Evapn. from 10' cm/28 days	Difference in Evapn. / the Std. Evapn. %
JAN.	30.4	22.8	33.3%
FEB.	21.75	17.65	23.2
MAR.	21.15	17.25	22.6
APRIL	14.4	11. 2	28.6
MAY	7.1	5.85	21.4
JUNE	5.65	4.1	37.8

Table A3 - 2 - Difference in evaporation from three similar evaporimeters at three different sites within 10 miles of each other.

SITE.	EVAPORATION TOTAL for 1948
Adelaide	64.3 inches.
Waite Institute	54.7 inches.
Dry Creek	82.9 inches.

Consequently, a great deal of investigational work has been done on empirically relating the evaporation rates measured from an evaporimeter with those from larger tanks and/or ponds. (Typical references: 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19, 23, 25, 34) Such empirical relations are shown to be practical and useful for the ponds and in the range of conditions under which they were tested.

Such relations have also been applied to systems on which they were not determined. That is the evaporation measured in some evaporimeter, corrected by such empirical relations, has been applied to other bodies of water or ponds. It must be emphasised however that if such a procedure is used:

(1) Only a rough indication of evaporation will be obtained and;

(2) A precise measurement of evaporation can be estimated only if the new vessel of water is identical to that used for the correlation and the exposure of it is identical.

(3) If the correlation is used outside the range of physical conditions under which the correlation was determined (e.g. temperature, humidity, wind velocity etc.) it is conceivable that the estimated evaporation may be considerably in error.

The most successful approach has been based on an energy balance, Cummings and Richardson (Ref. 20) stated:-

$$E = \frac{I - B - S - K - C}{L} \qquad \text{-----} \quad (3 - 1)$$

where E = evaporation rate

I = intensity of incoming radiation per unit area of pond surface.

B = block body radiation from the pond to the sky.

S = sensible heating of the pond.

K = a correction due to heat derived from or given up to the air as sensible heat.

C = combined correction for heat leakage through the walls and bottom, expansion of water and heat carried by flowing water.

H = I - B

L = latent heat of vaporisation.

All in consistent units.

Most of these terms can be evaluated in a straight forward manner. Bowen (Ref. 19) evaluated a ratio:

$$R = \frac{\text{Sensible heat swept away by the wind}}{\text{Latent heat carried away by vapour.}}$$

$$= \frac{K}{LE}$$

by assuming that the same mechanism exists for heat transfer and vapour transfer by eddy diffusion

$$R = 0.46 \frac{\theta_a - \theta_s}{p_a - p_s}$$

where θ_a = temperature of the air above the pond.

θ_s = temperature of the liquid surface.

p_a = partial pressure of water vapour in the air, m.m. of Hg.

p_s = Saturated vapour pressure of water at temperature t_s

Angstrom (Ref. 22) showed that for oceans R was small and could be neglected but more recent work has shown that R is not necessarily small (Ref. 20)

Hence $$E = \frac{H - S - LER - C}{L}$$

or $$I - B = H = S + C + LE\ (1 + R) \qquad \text{------} \quad (3 - 2)$$

I can be measured directly by a Pyrheliometer (Ref 23) and B can be assessed experimentally. Alternatively, H can be measured directly by means of a solarimeter (Ref. 15). For two water surfaces 1 and 2 receiving equal radiation I,

$$B_1 - B_2 + S_1 + LE_1 (1+R_1) + C_1$$

$$= S_2 + LE_2 (1+R_2) + C_2 \qquad \text{------} \quad (3 - 3)$$

Since water completely absorbs low temperature radiation, the water surface can be considered to radiate as a block body (Refs. 24, 19). If T_1 and T_2 are the absolute temperatures of the surfaces of the two ponds, $(T_2 - T_1)$ is small in the practical case (probably $\gtrsim 15^\circ C$) and T_1 and T_2 themselves are never great ($\lesssim 17^\circ C$)

$$\text{Also } B = \sigma T^4$$

$$\therefore \frac{dB}{dT} = 4\sigma T^3$$

which at 290° abs. $\simeq 4 \times (49.5 \times 10^{-10}) \ (290)^3$

$$\simeq 0.5 \text{ gm cal/cm}^2\text{/hr/}^\circ\text{abs}$$

Thus, approximately

$$B_1 - B_2 \simeq 0.5 \Delta T$$

and $$E_2 = \frac{0.5\,\Delta T + (S_1 - S_2) + (C_1 - C_2) + L\,(1 + R_1)\,E_1}{L\,(1 + R_2)} \qquad \text{----} \quad (3 - 4)$$

Thus if an evaporimeter and a large pond are calibrated to find C_1 and C_2 (respectively) and if E_1 be measured in the evaporimeter, E_2 can be

calculated for the pond from equation (3-4) if the surface, the bulk water and air temperatures and humidity are measured. It should be noted that the expression for R applies to instantaneous values only (Ref. 19) but in testing this expression average values were used.

Testing of this expression was carried out at Pasadena and Fort Collins in the U.S.A. (Ref. 20) with a small pan and a large pond for which C_2 was taken as zero i.e. the heat loss from the pond was taken as negligible (See § 2, Sec. (vi) and typical results are given in table (A3-3). For these results the heat budgets of the two bodies of water were compared and it is seen from these figures that confusion of the heat budgets is far more accurate than equating as equal the evaporations of the two bodies. This type of agreement is found in all the results in reference (20), page 531-2 figure 2 and 3. Thus equating the heat budgets of the evaporimeter and the pond will give a reasonable estimate of the pond evaporation from the measured tank evaporation. The main difficulty is in the measurement of the surface temperature θ_s which could be in considerable error, and many attempts have been made to find an expression which will eliminate θ_s (Refs. 4, 25) - See § 2.

It is clear then that an evaporimeter is a most unreliable "meter" for measuring the evaporation from another body of water. It has, however, very extensive use for measuring evaporation and rainfall (i.e. Gross evaporation) although it is accurate for neither (Ref. 23) since it is undoubtedly the easiest to operate and is the best method there is.

	September 1926		
DAY	20	21	23
Measured Pond evaporation LE_2	124.3	231.1	155.4
Sensible heat into pond S_2	-57.8	44.3	11.8
Convection heat K_2	40.1	-10.2	39.3
Heat budget for pond H_2	106.6	265.2	206.5
Measured evaporation from Evaporimeter LE_1	119.7	345.5	223.8
Sensible heat into evaporimeter S_1	-19.6	5.4	-58.1
Convection heat K_1	28.2	-46.0	41.7
$B_1 - B_2$	-41.5	-16.1	-75.4
Heat loss from Evaporimeter C_1	18.8	-14.6	33.7
Heat Budget H_1	105.6	274.2	225.7
$L(E_2 - E_1)$ & % of E_1	4.6 3.84%	-114.4 33.1%	-68.4 30.6%
$H_2 - H_1$ & % of H_1	1.0 0.95%	-9.0 3.3%	-19.2 8.5%

Table A3 - 3 - Typical results found at Pasadena for equation (3-4). All items are in heat units.

Calculation of Evaporation:

These limitations to the use of the evaporimeter as a measuring instrument were realised very early in the history of this field. Consequently, much work has been done to develop the mechanism of evaporation, its dependence on external physical conditions and generally to establish relations from which the evaporation can be calculated from a knowledge of other more easily measured meteorological variables, taken from easily standardised instruments. A good summary of the formula for evaporation up to 1915 is given by Meyer (Ref. 64).

In general, 3 main approaches have been used:

(i) the Empirical approach

(ii) the energy balance approach

(iii) the Aerodynamic approach.

(i) Empirical relations for calculating Evaporation.

The empirical approach has been based mainly on the work of Dalton (Ref. 8) who laid down the basic postulate, that water will evaporate if the vapour pressure of the water at the surface of the pond is greater than the partial pressure of water in the bulk of the air. Thus, the rate of evaporation is postulated to depend on a vapour pressure difference and a diffusion coefficient of the air layer, which in turn depends on the air velocity. This vapour pressure difference can be determined if the temperature of the surface and the saturated vapour pressure at that temperature are known, and the temperature and humidity of the air are known.

$$\text{Thus } E = K (p_s - p_a) \quad \text{-----} \quad (3\text{-}5)$$

and K = function of velocity u

Where E = evaporation rate

u = wind velocity at some given height above the pond surface.

p_s = saturated vapour pressure at θ_s

θ_s = surface temperature of the pond

p_a = partial pressure of water vapour in the air i.e. at θ_a

and K = Mass transfer coefficient for the system, postulated to be a function of velocity u.

Dalton did not develop this equation or his theory any further, but basing their work on this postulate, many other workers have attempted to evaluate k. The most notable achievements have been obtained by Rowher (Ref. 9) and Hefner (Ref. 10). Rowhers equation is:-

$E = 0.771\ (1.465-0.0186P)\ (0.44 + 0.118u)\ (p_s-p_a)$ inches of water/ 24 hours -----(3-6)

where p = mean barometic pressure ("Hg)

u_o = mean wind velocity (miles/hour) at the surface of the pond.

This factor can not be measured directly but is determined from the Meinzer curve for variation of wind velocity with height (Ref.11)

Hefner's equation is a simpler one to evaluate:-

$E = 0.085u_{20}\ (p_s - p_a)$ inches/month -----(3-7)

where u_{20} = velocity of the air at 20 feet above the water surface.

The general empirical relation then is:-

$E = i\ (p_s - p_a)\ (1 + gu)$ i and g are constants which must be evaluated for the particular body of water for which it is to be used.

If i and g are evaluated in an evaporimeter, the constants will apply only for that evaporimeter. It is impossible to determine i and g in an evaporimeter and then apply the equation to determine evaporation from a large pond or any other different body of water, for the reasons already give.

Further, there is a fundamental error in the concept of Dalton upon which these empirical relations have been based. Immediately above the water surface there is a thin laminar layer in which flow is laminar and mixing takes place by molecular diffusion. Above this layer is a zone of turbulent mixing, called the turbulent layer, in which the mixing process takes place by eddy diffusion. This depends upon the shearing stresses associated with the roughness of the ground, the wind velocity and density and wind structures. (Refs. 16, 17, 26, 36, 37, 44).

If water is evaporated into these regions from the water surface beneath, gradients of moisture are set up in the two layers, and providing both the tendency of mixing and the evaporation continue, are maintained. These gradients differ in the two layers, being linear in the laminar layer and far more complex in the turbulent layer, due to these different mechanisms of mixing. Thus no relation between the moisture concentration within the two layers, and hence the moisture gradient, can be written by a simple vapour pressure difference.

Further, the mass transfer coefficient can not be measured somewhere in the turbulent region only, since the mechanism of mass transfer differs in the two layers.

Hence it should be impossible to derive simple empirical constants to be used in combination with a vapour pressure difference of the liquid surface and the overlying air, and some wind velocity factor. There is such a difference in the mechanism of mixing, and the resulting moisture gradients in the two layers, that such empirical relations must be fundamentally in error.

Other empirical relations have been developed using different fundamental concepts. (Ref. 13)

Bigelow (Ref. 12) based his relation on vapour pressure gradients. Another basis was a "vapour pressure deficit" defined by the (saturated vapour pressure at some point) - (actual vapour pressure at the same point) (Ref. 18)

Brent (Ref. 26) Sutton (Ref. 41) & Pasquill (Ref. 42) give reference to an empirical relationship.

$$E = C\,(p_s - p^1_d)\,(u^{0.72}\, x_o^{0.88}\, y_o) \qquad \text{------ (3 - 8)}$$

for evaporation from a rectangle x_o long by y_o wide, where C is a constant, u is the wind velocity 2 meters above the water surface and p^1_d is the vapour pressure at a point unaffected by evaporation.

Many of these relations, however, are unrelated to the physical principles of evaporation and can not be expected to yield satisfactory results. A practical difficulty in the use of all these

empirical relations is to evaluate the "wind velocity" which varies with height and with time. (Ref. 12)

The D.S.I.R. (N.Z.) tested some of these empirical relations and found only poor agreement between actual and calculated evaporations. (Refs. 11 and 14)

The Hefner formula was found to be as accurate as the Rowher, easier to use and generally the best empirical relation tested. Throughout the literature however, it is apparent that empirical relations do not give good results (Ref. 16) and that these relations, depending upon constants experimentally determined in an evaporimeter, or its equivalent are applicable to that one vessel only, and even then, are inaccurate for that vessel over any practical range of evaporation.

(ii) Energy balance Method.

Bowen (Ref. 19) has stated the basic postulate for the energy balance method of determining evaporation.

> "The process of evaporation and diffusion of water vapour into a body of air is exactly equivalent to the conduction, or diffusion of specific heat energy into the body of water"

Thus, the weight of water evaporated,

$$= \frac{\text{Heat energy absorbed by the fluid}}{\text{Latent heat of evaporation}}$$

The energy balances set up in § (2) and given as Equation (2-6) clearly contain two unknowns which can not be measured easily - the surface temperature θ_{p_s} and the evaporation rate E. The use of the energy balance to determine E, requires the elimination of θ_{p_s} from this equation.

This has been the basis of much investigational work by Bowen (Ref. 19) Cummings and Richardson (Ref. 20) and later Penman (Ref. 21), Ferguson (Ref. 25), Bonython (Refs. 4 and 23) and the D.S.I.R. (Ref. 14) and many others.

In all these approaches, θ_{p_s} has been eliminated, or calculated, by using some other empirical relation in combination with the energy balance equation.

Generally a relation for E in terms of p_s, which itself depends on θ_{p_s}, is substituted into equation (2-6) to get a relation connecting θ_{p_s} and p_s. This can be solved if the vapour pressure of the brine versus temperature plot is known and then E can be calculated.

Ferguson (Ref. 25) reduced equation (2-6) to the form:-

$$\frac{dI_{abs}}{dt} = \rho_p \, hS_p \frac{d\theta}{dt} + 2h_c \left[\zeta(\theta) - p_a \right] + h_c \left[\theta - \theta_a \right]$$

where θ = temperature of the pond surface

$\zeta(\theta) \equiv p_s$: given as the plot of staurated vapour pressure versus temperature

$E = 2h_c (p_s - p_a)$ (Dalton type)

Ferguson assumed a curve for $\frac{dI_{abs}}{dt}$ against the time of day, $\eta(t)$, and for the air temperature, a diurnal variation equal to $\xi(t)$. For h and p_a constant:

$$\eta(t) = \rho_p \, h \, S_p \frac{d\theta}{dt} + 2h_c \left[\zeta(t) - p_a \right] + h_c \left[\theta - \xi(t) \right]$$

where $\eta(t)$, $\xi(t)$ are functions of time given as plots. This equation was solved using a differential analyser to give continuous values of θ_{p_s} for various starting values, expressable then as:-

$$\theta_{p_s} = f(t)$$

Evaporation could then be obtained from :-

$$E = \frac{2h_c}{L} \left[f(t) - p_a \right]$$

Penman (Ref. 21) used what was called a "sink strength" method. He showed that :-

$$E = \frac{H\Delta + E_a \gamma}{\Delta + \gamma} \qquad \text{----- (3-9)}$$

where E_a = is the evaporation calculated byputting p_a in place of p_s in the equation $E = (p_s - p_d) f(u)$, p_d = vapour pressure at the dew point.

$$H = R_c (1 - r - \mu) - \sigma T_a^4 (0.56 - 0.92 \sqrt{p_d}) (1 - 0.09m)$$

R_c = the measured short wave radiation

r = the reflection coefficient for the water surface

By measurement,

$\underline{r}$ = 0.06 for water

= 0.10 for bare soil

= 0.20 for turf.

μ = fraction of R_c used in photosynthetic processes. Is zero for water surfaces.

$\frac{m}{10}$ = The fraction of the sky covered by the cloud = $1-\frac{n}{N}$

$$= 1 - \frac{\text{actual hours of sunshine}}{\text{possible hours}}$$

Δ = The slope of the vapour pressure - temperature curve at $\theta = \theta_a$

γ = constant in Bowen's equation $R = \gamma \frac{(\theta_s - \theta_a)}{(p_s - p_a)}$

This expression was tested experimentally at the meteorological enclosure at Rothamsted, but very poor results were obtained. (Ref. 21, Figs. 2, 3, 5)

The relation still includes the mass transfer coefficient as a function of wind velocity, a Dalton type postulate of $E = f(u)\ (p_s - p_d)$ and so, in fact, is only an empirical relation "patched up". It would not be expected to give any better result than the truly empirical relation.

Bonython (Ref. 4) discusses a very similar approach developed by Ferguson (Ref. 25) in which they have solved for the surface temperature θ_s (and hence p_s) by combination of Dalton's equation and the energy balance.

For the energy balance they pointed out that the gain of energy by the pond can be measured by a solarimeter (Ref. 15) or calculated by the methods of Brent (Ref. 26) or Penman (Ref. 21) and that the loss of energy can be calculated by assuming that, for this term only, the surface temperature equal to the air temperature (Ref. 20) after a connection, derived by Callender (Ref. 25) had been applied.

Assuming θ_s constant, and no heat loss to the surroundings by conduction:-

if Q = nett gain of radiant energy, assuming $\theta_a = \theta_s$

i.e. $Q = Lk\ (p_s - p_a) + h_c(\theta_s - \theta_a)$

h_c = heat transfer coefficient

e.g. $= 0.48 + 0.083\ \mu$ (Ref. 25)

$\frac{h_c}{Lk} = 0.5$ for water (Ref. 67, 45)

$\therefore\ 2p_s + \theta_s = \frac{Q}{h} + 2p_a + \theta_a$ ——— (3-10)

Allowing for long wave radiation of energy from the pond -

$2p_s + \theta_s\ (1+\frac{\omega}{h_c}) = \frac{Q^1}{h} + 2p_a + \theta_a(1+\frac{\omega}{h_c})$ —— (3-11)

where $\omega = \sigma\ T_a^4$

$Q^1 = 2h_c\ (p_s - p_a) + h_c^1\ (\theta_s - \theta_a)$

$h_c^1 = h_c + \omega$

Plotting equation (3-11) on the curve of vapour pressure versus temperature (as in Fig. A3.4), the value of p_s and θ_s can be read off.

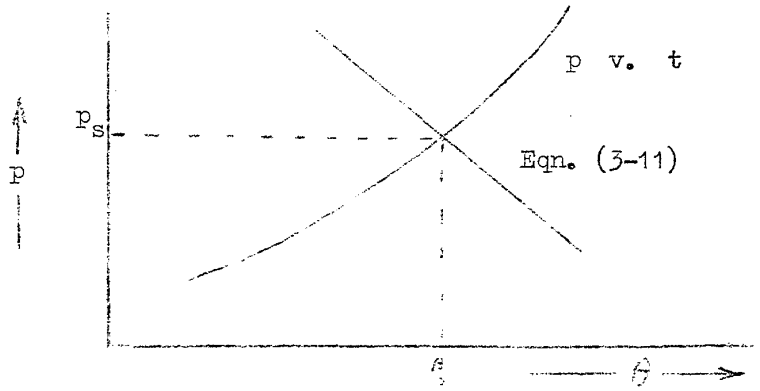


Figure (A3.4) - Solution of equation (3-11)

From these values of θ_s and p_s, the evaporation can be calculated from -

$$E = \frac{2h_c}{L}\ (p_s - p_a) \qquad \text{—— (3-12)}$$

The correction of Callender (Ref. 25) results in breaking up Q so that-

$Q_p + \sigma y T_a^4 - \sigma T_s^4 = 2h_c(p_s - p_a) + h_c(\theta_s - \theta_a)$

Q_p = Shortwave radiation absorbed

y = Brunt's Coefficient for long wave radiation. (Ref. 26)

$$\text{i.e. } Q_p - \sigma T_a^4 (1-y) = 2h_c (p_s - p_a) + (\theta_s - \theta_a)(h_c + 4\sigma T_a^3) \quad \text{------(3-12)}$$

This expression, for E, can be evaluated again using Dalton's equation. The heat transfer coefficient, h_c, is dependent on the wind velocity "near" the surface of the pond which is most difficult to evaluate. The most serious drawback is the variation of all the data with time which means that average values must be used. It would appear that there is no reason why this expression should be any more accurate than the original empirical relations and Bonython, when testing this expression, showed that values were up to 20% low for a standard evaporimeter and up to 6.5% for a 10 foot diameter tank. (Ref. 4, page 207, 208 Table IV and Fig. 2)

The D.S.I.R. made tests based on a very similar approach (Ref. 14) and attempted to derive a correlation between evaporation from pure water in an evaporimeter, and calculated, based on this method, values for brine evaporating in large ponds. They found no workable correlation and generally their results were not satisfactory.

Bonython has tried to justify the use of such equations by applying measured values which suit the equations. It is clear that such a procedure has little practical value.

The energy balance in itself is exact and its modification, by combining it with empirical relations which are grossly in error, does not seem justified.

The energy balance approach, then, will never produce a satisfactory relation for calculating the evaporation while it is to be combined with empirical equations, based on vapour pressure differences in the laminar and turbulent layers, wind velocity factors for heat or mass transfer and arbitrary constants determined in an evaporimeter. The inherent errors, due to fundamental mistakes in the postulate, of such empirical equations will prohibit any better results being obtained by energy balances.

(iii) Aerodynamic approach.

For evaporation to be initiated and to continue, it is not sufficient that energy should be supplied to the brine, but some mechanism must exist whereby vapour is removed from the surface of the pond to prevent the recapture of the water molecules.

An evaluation of this mechanism was suggested as a possible method of calculating the evaporation from a surface and considerable interest was taken in the study of the turbulence and intensity of turbulent mixing which was being undertaken.

The recent advances in aviation stimulated this study of the lower atmosphere and in particular the turbulent layer (Ref. 17). As mentioned above, the mixing processes in the two layers above the surface differ considerably, molecular diffusion being the mechanism in the laminar layer, and eddy diffusion in the turbulent layer. The latter is stated to be 25,000 times more effective than molecular diffusion for mixing processes (Ref. 16) so that it was natural for the turbulent layer to be more closely studied.

These mixing processes cause a transfer of mass in a vertical direction (Ref. 17) and the tendency towards equalising the air velocities at various heights above the surface depends on the rate of vertical mass interchange - the intensity of turbulent mixing.

Moisture added to this air at the surface will be transferred vertically. Thus a moisture gradient will be established above an evaporating surface which will remain only as long as evaporation continues, the moisture being distributed evenly if evaporation ceases. The greater the turbulent mixing, the greater will be the tendency to establish a uniform moisture content or, the greater must be the evaporation to maintain a constant moisture gradient. That is, the greater the moisture gradient for a given intensity of turbulent mixing, the greater will the evaporation need to be to maintain it, so that the rate of evaporation should be calculable by considering the moisture gradient and the intensity of turbulent mixing in the turbulent layer.

This basis was postulated by Schmidt (Ref. 27) as -

$$E = -A\frac{dq}{dZ} \qquad \text{---- (3-13)}$$

A = Coefficient of turbulent mixing or the Austausch Coefficient.

$\frac{dq}{dZ}$ = Rate of change of moisture concentration w.r.t. height above the evaporating surface.

The important investigational work has been to determine A Schmidt (Ref. 27) first postulated the nature of A and many workers have followed his lead (References 29-39) to give -

$$A = \rho_a (k_o Z)^2 \cdot \frac{d\bar{u}}{dZ}$$

$$= \rho_a k_o^2 Z (\bar{u}_2 - u_1) / \ln \frac{Z_2}{Z_1} \qquad \text{---- (3-14)}$$

Where $\bar{u}_1$ = mean velocity of air in a direction parallel to the ground at a height Z, above the surface etc.

k_o = von Kármán's universal turbulence constant, related to mixing length l^* by $l^* = k_o Z$ (Ref.37)

ρ_a = air density

The evaporation equation then follows directly (Refs. 37, 38, 39 particularly) as -

$$E = \frac{833\, k_o^2 (p_1 - p_2)(\bar{u}_2 - \bar{u}_1)}{(0 + 459.4)\left(\ln \frac{Z_2}{Z_1}\right)^2} \text{ inches/hour.}$$

---- (3-15)

k_o has been measured in wind tunnels and for water flowing in pipes and open channels. It varies from about 0.22 to 0.43. It has never been measured for the atmosphere due to the difficulty of evaluating the shear forces, a value of 0.38 usually being taken. (Refs. 37 and 38). Ertel (Ref. 35) has developed a method of calculating A from short time fluctuations of any physical property of the atmosphere such as temperature, humidity, momentum, etc. from which k_o can be determined.

Leighly (Ref. 73) applied a similar approach to the laminar layer, but found the moisture gradient impossible to measure practically because of the small thickness of this layer. It would clearly be impossible on a rippled surface.

Sverdrup (Ref. 39) used similar relations for making calculations over oceans.

To determine evaporation by this aerodynamic approach requires the measurement of moisture concentration (e.g. humidity or partial pressures) and wind velocities at two levels within the turbulent layer to give the moisture gradient and intensity of turbulent mixing.

The practical application has been fully discussed by Bonython (Ref. 15) and by others (References 16 and 40). Bonython shows that great accuracy is required in determining the moisture content and suggests that continuous records should be taken. In Dry Creek, South Australia, humidity differences of the order of one millimeter of mercury in 26 feet (Z_2= 28', Z_1 = 2') was observed.

The equations derived are rigorously correct for an adiabatic atmosphere, but for cases of thermal stability a correction must be applied. Over a day, evaporation under thermally stable conditions is small compared with that under adiabatic conditions. For condensation of dew, where turbulence is always negligible, however, serious error may result if the correction is not applied. (Ref. 46) Certain critical values of temperature and wind structure, such as light winds and temperature inversions, are found to suppress the effects of turbulent mixing so that moisture transfer is by molecular diffusion alone and error in the use of these equations may result. (Ref. 47)

It seems that this approach will give the most accurate estimation of evaporation rates from a body of water and some investigational work will have to be done to determine the feasibility of the method for evaporation from brine ponds, where these brine ponds vary in concentration for a system.

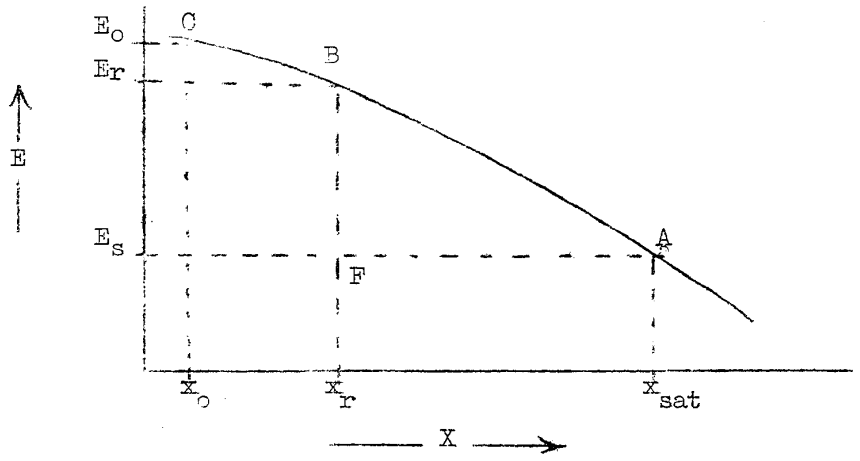
It is possible that the evaporation from one pond may affect the atmosphere above others, and as a result the evaporations calculated for neighbouring ponds would be in error. It seems probable that Bonython (Ref. 15) or Thornthwaite and Holzman (Ref. 16) will test for such interference. In an established salt works only one record would be needed, probably over a large expanse of water, and predetermined

factors applied to give the evaporation rate for any concentration brine at any time. Here the interference would not be important but in the determination of the evaporation factors, it would have to be considered.

With such a relation, it would be possible to establish accurate continuous plots of evaporation from large ponds of brine against the concentration of brine, for a number of years prior to the completion of the design and construction of the salt works. In view of the work described in this paper such investigation appears essential.

Appendix for § 6.

Consider the plot of evaporation rate versus concentration as in the accompanying figure.



If there is only one pond in the system, its concentration must always be x_{sat} so that $E = E_s$ is a minimum and the output a minimum. For 2 ponds, with x_1 = say x_r the evaporation rates are E_r ($> E_s$) and E_s and the output will be an improvement on one pond.

For an infinite number of ponds, the total evaporation from the system is clearly a maximum and the output a maximum also.

APPENDIX

Exact solution of Equation.

For the solution after infinite time, S.C.J. Smith showed that if-

$$x = a_o + a_1 \sin t + a_2 \sin^2 t + \text{--------}$$

$$+ \cos t \,(b_o + b_1 \sin t + b_2 \sin^2 t$$

the order of the coefficients was, for h = 12",

$$a_o = +0.05$$

$$b_o = +0.01$$

$$a_1 = -0.001$$

$$b_1 = -0.001$$

and that the other coefficients, a_2, b_2 etc. could be neglected.

By a trial and error method he showed that

$$x = 0.0454_5 - 0.00035_1 \sin t$$

$$+ \cos t \,(0.0028_3 - 0.00007_4 \sin t)$$

from which the following values were calculated.

VALUES FOR POND CONCENTRATION X, AT DIFFERENT TIMES, IN THE EQUATION: $x = 0.0454_5 - 0.00035 \sin t + \cos t\,(0.0028_3 - 0.000007_4 \sin t)$			
t	x	t	x
0	$+0.0482_8$	π	$+0.0426_2$
$\pi/12$	0480_7	$13\pi/12$	0427_9
$\pi/6$	0476_9	$7\pi/6$	0431_5
$\pi/4$	0471_6	$5\pi/4$	0436_6
$\pi/3$	0465_4	$4\pi/3$	0443_0
$5\pi/12$	0458_2	$17\pi/12$	0450_4
$\pi/2$	0451_0	$3\pi/2$	0458_0
$7\pi/12$	0444_0	$19\pi/12$	0465_4
$2\pi/3$	0437_6	$5\pi/3$	0472_0
$3\pi/4$	0432_4	$7\pi/4$	0477_4
$5\pi/6$	0428_5	$11\pi/6$	0481_1
$11\pi/12$	04265	$23\pi/12$	0482_9
		2π	0482_8

These values were plotted on figure (6.13) and show a discrepancy of less than $\frac{1}{2}\%$ from the values obtained by the differential analyser.